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An Integrability Primer for the Gauge-Gravity Correspondence: an Introduction

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Abstract

We introduce a series of articles reviewing various aspects of integrable models relevant to the AdS/CFT correspondence. Topics covered in these reviews are: classical integrability, Yangian symmetry, factorized scattering, the Bethe ansatz, the thermodynamic Bethe ansatz, and integrable structures in (conformal) quantum field theory. In the present article we highlight how these concepts have found application in AdS/CFT, and provide a brief overview of the material contained in this series.

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In this article we introduce a series of articles reviewing aspects of integrable models. The articles provide a pedagogical introduction to the topic of integrability, with special emphasis on methods relevant in the AdS/CFT correspondence. After a brief motivation regarding the value of general integrable models in the development of theoretical physics, here we discuss the application of the framework of integrability to the AdS/CFT correspondence. We then provide an overview of the material contained in the various reviews, referring back to AdS/CFT applications, and indicating links between the reviews themselves and to the relevant literature. While written with an AdS/CFT background in mind, the methods covered in the reviews themselves have applications throughout the wider field of integrability.

Integrability

Integrable models appear throughout theoretical physics, starting from classical mechanics where models such as the Kepler problem can be solved—in the sense of the Liouville theorem—by integration. In general, integrable models show special behaviour due to many underlying symmetries, symmetries due to which they can often be exactly solved. Only a fraction of the physical systems appearing in nature can be described in these terms. Nevertheless, integrable models offer insight into real-world situations through universality, or when used as a theoretical laboratory to develop new ideas. In statistical mechanics for example, many subtleties of the thermodynamic limit have been understood by working out specific models, notably phase transitions in the Lenz-Ising model and the role of boundary conditions in the ice model. In hydrodynamics, the Korteweg-de Vries equation illustrates how a non-linear partial differential equation can admit stable, wave-like localized solutions: *solitons*. In condensed matter physics, both integrable quantum spin chains and one-dimensional gases of almost-free particles play a pivotal role. Finally, in quantum field theories (QFTs) in two space-time dimensions, exactly solvable models helped unravel phenomena like dimensional transmutation, as in the case of the chiral Gross-Neveu model, or concepts like bosonisation, as in the case of the sine-Gordon and Thirring models. The general framework to study such integrable QFTs, mainly associated to *inverse* and *factorized scattering*, was laid down in the 1970s and has found numerous applications since.

Integrability in AdS/CFT

In recent years, the general framework of integrability has been successfully applied in the context of the AdS/CFT correspondence [1], a concrete realisation of the holographic principle [2]. According to this correspondence, string theory on anti-de Sitter (AdS) backgrounds is dual (equivalent) to conformal quantum field theory (CFT) on the "boundary" of AdS. The

canonical example of this duality is the correspondence between closed type IIB superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions. Both sides of this duality can be studied using integrability-based techniques, at least in the so-called planar limit. Similar ideas apply to lower dimensional AdS backgrounds, as well as to deformations of these backgrounds. Part of this progress is reviewed in e.g. [3, 4, 5, 6, 7, 8].

In the AdS/CFT context, integrability enters naturally on the string theory side as a property of particular two dimensional field theories.[1] The details of two dimensional field theories can be such that their classical equations of motion can be tackled by the *inverse scattering method*, an approach initiated in the 1960s [9] and mainly developed in the following decade [10]. In particular, the equations of motion of such integrable field theories can be represented as the flatness of a so-called *Lax connection*. Now in the planar limit, closed string theory reduces to field theory on a two-dimensional cylinder—the worldsheet of a single string—and this field theory is integrable [11] in the above sense. By expanding the machinery of classical integrability it is possible to tackle the semi-classical spectrum of integrable field theories as well, resulting in what is known as *finite-gap equations* [12]. The semi-classical limit of our closed string can be approached in this spirit [13], see also [14]. Moving beyond the semi-classical spectrum is more involved, as will come back shortly.

The integrability appearing on the CFT side of AdS/CFT is that of integrable spin chains. Integrable spin chains such as the Heisenberg spin chain can be solved by the *Bethe ansatz* [15]. This is an ansatz for the eigenfunctions of a spin-chain Hamiltonian, written in terms of collective excitations called magnons or spin waves, and their scattering matrix (S matrix). There is an underlying algebraic structure however, based on an *R matrix* and *Lax matrix*, which leads to the algebraic Bethe ansatz also known as the *quantum inverse scattering method* [16]. Historically, the appearance of an integrable spin chain was the first indication of integrability in AdS/CFT [17].[2] Working in $\mathcal{N} = 4$ SYM theory at one loop order, Minahan and Zarembo showed [17] that by identifying single-trace operators with particular spin-chain states, the dilatation operator—whose eigenvalues yield the anomalous dimensions of such operators—becomes the Hamiltonian of an integrable spin chain [19], whose spectrum can be found via the Bethe ansatz.

Though different in their appearance, these two types of integrability share a common symmetry structure. Two-dimensional integrable field theories typically have infinitely many conserved charges that can be packaged into a powerful algebraic structures, and these same structures come back in spin chains. A prototypical example of such a structure, particularly important in AdS/CFT, is the Yangian algebra [20]. Indeed, the classical integrability of the

[1] Our exposition is not chronological. Some historical aspects are discussed in [4].

[2] In the context of gauge theory, integrability was previously encountered in high energy hadron scattering in QCD [18].

string brings with it an infinite set of conserved charges [11], which form a Yangian algebra that can also be seen as the symmetry of the quantum spin chain of SYM [21].

Going beyond one loop in SYM, or semi-classics in string theory, requires more work, but is possible. The presence of integrable structures at higher loops in SYM [22, 23, 24] makes it possible to find an exact S matrix for the spin chain magnons [24] and write down an *asymptotic Bethe ansatz* [23]. This S matrix and Bethe ansatz have counterparts in the dual string theory [25, 26, 27]. There, the S matrix is simply the worldsheet S matrix of the light-cone gauge-fixed string [27]. To define this S matrix and the associated asymptotic states we need to take the limit where the length of the gauge-fixed string (volume of the theory) goes to infinity. The asymptotic Bethe ansatz then arises by re-imposing periodic boundary conditions on approximate wavefunctions obtained from the S matrix using the ideas of *factorized scattering*.

The reason for distinguishing the asymptotic Bethe ansatz from the (exact) Bethe ansatz is clear on the string theory side: merely imposing periodic boundary conditions while working with the S matrix of the infinite length string, neglects possible virtual particles wrapping around the worldsheet (cylinder) [28]. This failure of the asymptotic Bethe ansatz for the string [29] is paralleled by a similar breakdown in the $\mathcal{N} = 4$ SYM spin-chain [30]. Here the dilatation operator features interactions whose range increases with the loop order, so that eventually the interaction range is of the order of the length of the composite operator under consideration, and the Bethe ansatz breaks down. In relativistic field theory models these *finite size effects* can be understood by integrability techniques [31, 32] using the *thermodynamic Bethe ansatz* [33]. Extending these ideas to the integrable string sigma model gives the $\mathrm{AdS}_5/\mathrm{CFT}_4$ thermodynamic Bethe ansatz [34] and its improvement known as the *quantum spectral curve* [35]. It is now possible to compute the energy of closed string states nonperturbatively with arbitrary numerical precision, or analytically in a weak coupling expansion up to loop orders prohibitively difficult to reach by conventional techniques.

The chain of reasoning leading up to this description of the spectral problem involves various unproven albeit well-tested assumptions, in particular the hypothesis that integrability persists at the quantum level at arbitrary coupling. In some simpler models—specific conformal field theories and their massive deformations—the resulting structures can be more rigorously derived from first principles by methods introduced by Baxter [36] and developed by Bazhanov, Lukyanov and Zamolodchikov [37]. Doing so in the AdS/CFT context would undoubtedly provide remarkable insights, but thus far the answer appears to be elusive [38].

The articles

Beyond the spectral problem for AdS_5/CFT_4 described above, integrability based approaches to other observables and other instances of AdS/CFT are being actively pursued. While this landscape is motivation for this series of articles, we do not aim to review all of it. Rather, we review some of the key ideas upon which the progress in this field is based, ideas which can often be introduced and understood in simpler models. In fact, these ideas and methods are central to many integrable models and not just those appearing in AdS/CFT, and as such the material presented in this series is relevant to integrability in general.

Our aims have required us to make choices: we will cover a relatively broad set of techniques and highlight how they are related to each other, at the expense of the details of the many models where they can be applied. Our key example is the chiral Gross-Neveu model, as a good compromise between keeping relevant features of general integrable models and reducing technical complications. Where appropriate, the individual chapters contain further references to the AdS/CFT or integrability literature.

Below we give a detailed overview of each of the chapters, appearing in their suggested reading order [39, 40, 41, 42, 43, 44]. The lectures on Classical Integrability and Yangian Symmetry give the historical and mathematical background for the other lectures. Afterwards we turn our attention to scattering matrices, with special focus on integrable scattering in two dimensional QFTs. Building on this, we discuss how to obtain (asymptotic) Bethe ansatz equations, both in the original and algebraic approach. Next, we move to the thermodynamic Bethe ansatz as a tool to describe integrable models either at finite temperature or at finite size. The last article explores the relation between the symmetries stemming from integrability and those of conformal symmetry in two-dimensional QFTs, tying together most of the material presented in the previous articles. We have aimed to keep notation uniform throughout the articles.

Chapter I: Classical Integrability. The chapter on Classical Integrability [39] deals with classical Hamiltonian systems which are integrable by Liouville's theorem, and explores the algebraic techniques which are available to exactly solve such systems. This part is mostly concerned with the classical inverse scattering method, where Lax pairs and r-matrices are treated and their properties outlined, culminating in a discussion of soliton solutions and the Gel'fand-Levitan-Marchenko equation. Although most of the these topics are reviewed in standard monographs, such as [45], some of the algebraic aspects—such as the Belavin-Drinfeld theorems [46]—tend not to be, and are here presented in a compact uniform fashion.

This section introduces tools of classical integrability that play an important role in describing strings on various anti-de Sitter spaces [4, 6]. Although the applications to string

theory are rich of algebraic complications (see for instance [47, 7]), the basic ideas are practically the same as those contained in this review, which therefore works as a good entry point for anyone interested in delving into the modern topics connected to AdS/CFT integrability in the strong coupling (classical string theory) regime.

Chapter II: Yangian Symmetry. The quantum Yang-Baxter equation represents one of the most prominent features of integrable models. The lectures on Yangian symmetry of this chapter [40] deal with the algebraic structure that underlies rational solutions to this equation. The Yangian beautifully extends the concepts of classical integrability reviewed in Chapter I [39]. Mathematically, this symmetry enhances an ordinary Lie algebra to a so-called *quantum group* with the structure of a Hopf algebra.[3] In physical models, the crucial difference of the Yangian to ordinary Lie algebra symmetries lies in the fact that the Yangian generators represent *non-local* symmetries, which act on a discrete or continuous space. This one-dimensional space can be realized in many different ways making the Yangian a rather universal concept with strong implications for a given theory—classical or quantum.

In particular, the Yangian appears in the context of (1+1)-dimensional field theories, in spin chain models and it underlies the integrability of the AdS/CFT correspondence. Its prime application is to bootstrap integrable scattering matrices which are discussed in Chapter III [41], and its algebraic structure provides the basis for the Bethe ansatz reviewed in Chapter IV [42]. Keeping an eye on the historic development, we provide an introduction to the subject that contains both the more mature discussions of Yangian symmetry in two-dimensional models (see e.g. [48]), as well as its modern application to the gauge/gravity duality (see e.g. [49]). Generic definitions and concepts are illustrated by means of examples including the two-dimensional chiral Gross-Neveu model, the Heisenberg spin chain, as well as $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. These lectures aim at providing an introductory overview, which draws connections between different physical applications and mathematical aspects of the rich subject of Yangian symmetry.

Chapter III: S matrices and Integrability. The third chapter [41] of this collection deals with a fundamental object in quantum integrable theories: the S matrix, *i.e.* the operator that maps initial to final states in a scattering process.

First of all, knowing the S matrix is crucial for calculating the energy spectra in the large volume limit, via the derivation of the asymptotic Bethe ansatz, as will be reviewed in Chapter IV [42]. Beyond the asymptotic regime, the S matrix is a key ingredient for the leading and exact finite-size corrections of the energies, calculated by the Lüscher formulas and the thermodynamic Bethe ansatz respectively, both discussed in Chapter V [43].

[3]The Yangian represents one member of the family of quantum groups that are found in integrable models.

The miracle happening in (1+1)-dimensional integrable models, as we will explain in this chapter [41], is the possibility to determine the S matrix exactly, due to the highly constraining conservation laws of these particular theories and few analytical assumptions. This also makes it possible to derive the S matrices for bound states, if any, of the theory.

From an algebraic point of view, the S matrix can often be identified with a representation of the universal R-matrix of a Hopf algebra, reviewed in Chapter II [40]. This places the properties of the S matrix in an algebraic setting, and allow us to generalize its derivation also beyond the relativistic case.

Finally, the role played by the S matrix in the determination of form factors will also be briefly mentioned, and the S matrices of sine-Gordon, SU(2) and SU(3) chiral Gross-Neveu models will be discussed. Through these examples, it will be possible to show how to derive the exact S matrices and some simple form factors in practice, both for fundamental and bound states. We will also briefly discuss non-relativistic S matrices and overview their applications to the AdS/CFT correspondence.

Chapter IV: The Bethe Ansatz. Bethe ansatz techniques originated from the exploration of spin chains as models of condensed matter systems. The same methods also turned out to play a key role in computing the spectrum of 2d integrable field theories. These two applications have been recently united in the context of integrability in the AdS/CFT correspondence. The Bethe ansatz in AdS/CFT [23] realizes a beautiful interpolation between integrable spin chains on the gauge theory side [17] and the integrable structure of a 2d sigma model on the string theory side [25].

Chapter IV of this collection [42] covers various aspects of the Bethe ansatz in a pedagogical way, serving as a preparation for understanding its applications in AdS/CFT. This chapter logically continues the article dedicated to exact S matrices [41]. It is discussed how, knowing the S matrix, we can use the Bethe ansatz to find the theory's *non-perturbative* spectrum, albeit only in large volume. As explicit examples, the two-dimensional SU(2) and SU(3) chiral Gross-Neveu models are considered. We will see that to compute the spectrum one should first solve an auxiliary spin chain, which in these cases is the famous Heisenberg XXX model. Its solution is covered in detail, including the coordinate and the algebraic Bethe ansatz approaches, as well as the nested Bethe ansatz in the SU(3) case. It is also demonstrated that in the classical limit the Bethe equations encode a Riemann surface known as the spectral curve of the model. Finally, it is shown how the familiar 1d oscillator in quantum mechanics can be solved via a Bethe ansatz-like method.

Chapter V: The Thermodynamic Bethe Ansatz. The thermodynamic Bethe ansatz (TBA) is a method used to describe the thermodynamics of integrable systems solved by

the Bethe ansatz, resulting in a set of integral equations whose solution determines the free energy of the model in thermodynamic equilibrium. After its inception at the end of the sixties by Yang and Yang [33] to describe the thermodynamics of the one dimensional Bose gas with delta function interaction (the Lieb-Liniger model), the TBA was quickly and broadly adopted. Its use now ranges from describing the thermodynamics of integrable spin chain models such as the XXZ spin chain [50], to computing the spectra of integrable field theories on circles of finite circumference [31, 32] and beyond. This is how the TBA originally entered in the AdS/CFT correspondence for instance: the exact energy spectrum of the $AdS_5 \times S^5$ superstring is encoded in a set of TBA equations [34]. At the same time, equations of TBA type arise in determining the area of classical string worldsheets [51] for example.

The fifth chapter of this series [43] provides an introduction to the TBA, focussing on the conceptual ingredients—root distributions, counting functions, particle and hole densities, the string hypothesis in case of bound states—that underlie this method. We illustrate this discussion on concrete examples, starting from simple free electrons, then the original Bose gas, and finally the XXX spin chain and SU(2) chiral Gross-Neveu model as respectively spin chain and field theory examples with nontrivial string hypotheses. We also discuss the simplification of TBA equations, the derivation of Y systems from TBA equations and the equivalence between the two modulo analyticity data, and the use of the TBA in finite volume integrable field theory, including excited states and Lüscher formulae.

Chapter VI: Integrable Structures in Quantum Field Theory. The expression "integrable structures" appearing in the title of this article can be interpreted in two different ways. On the one hand, it is used as a label for fundamental objects appearing in quantum integrable models, that is to say integrals of motion, transfer matrices, Baxter Q-operators and so on. There exists, however, a broader meaning to this expression, referring to the nature and the properties of the algebraic foundations on which the quantum integrable theories stand. The fundamental objects named above then appear as the main characters in the story of the integrable structures. This tale has been known for decades in the case of spin chains and lattice models [36], but it was only in the nineties that it was first told for a 2D quantum field theory [37, 52]. The approach of Bazhanov, Lukyanov and Zamolodchikov, nowadays referred to as the BLZ method, was the first successful attempt at the construction of the fundamental integrability objects from the algebraic structure of a field theory. Although this does not deal directly with theories associated to sigma models and AdS/CFT, it nonetheless provides general recipes with broader applications [32, 53]. Another important, pedagogical aspect, is that the BLZ method employs an array of mathematical concepts with connections to most approaches to integrability. In this way, the sixth chapter of this

series [44] can serve as a playground where the methods and concepts discussed in the other chapters can be put in motion.

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References

[1] J. M. Maldacena, *"The large N limit of superconformal field theories and supergravity"*, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200. • E. Witten, *"Anti-de Sitter space and holography"*, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150. • S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"Gauge theory correlators from non-critical string theory"*, Phys. Lett. B428, 105 (1998), hep-th/9802109.

[2] G. 't Hooft, *"Dimensional reduction in quantum gravity"*, gr-qc/9310026. • L. Susskind, *"The World as a hologram"*, J. Math. Phys. 36, 6377 (1995), hep-th/9409089.

[3] N. Beisert, *"The dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory and integrability"*, Phys. Rept. 405, 1 (2004), hep-th/0407277.

[4] G. Arutyunov and S. Frolov, *"Foundations of the* $\mathrm{AdS}_5 \times S^5$ *Superstring. Part I"*, J.Phys.A A42, 254003 (2009), arxiv:0901.4937.

[5] N. Gromov, *"Integrability in* AdS/CFT *correspondence: Quasi-classical analysis"*, J.Phys.A A42, 254004 (2009). • D. Serban, *"Integrability and the AdS/CFT correspondence"*, J. Phys. A44, 124001 (2011), arxiv:1003.4214.

[6] N. Beisert et al., *"Review of AdS/CFT Integrability: An Overview"*, Lett.Math.Phys. 99, 3 (2012), arxiv:1012.3982.

[7] A. Sfondrini, *"Towards integrability for* $\mathrm{AdS}_3/\mathrm{CFT}_2$*"*, J.Phys. A48, 023001 (2015), arxiv:1406.2971.

[8] S. J. van Tongeren, *"Integrability of the* $\mathrm{AdS}_5 \times S^5$ *superstring and its deformations"*, J.Phys. A47, 433001 (2014), arxiv:1310.4854.

[9] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *"Method for solving the Korteweg-deVries equation"*, Phys. Rev. Lett. 19, 1095 (1967).

[10] L. D. Faddeev and V. E. Zakharov, *"Korteweg-de Vries equation: A Completely integrable Hamiltonian system"*, Funct. Anal. Appl. 5, 280 (1971), [Funkt. Anal. Pril.5N4,18(1971)]. • V. E. Zakharov and A. B. Shabat, *"A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I"*, Functional Analysis and Its Applications 8, 226, http://dx.doi.org/10.1007/BF01075696. • V. E. Zakharov and A. B. Shabat, *"Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II"*, Funct. Anal. Appl. 13, 166 (1979), [Funkt. Anal. Pril.13N3,13(1979)].

[11] I. Bena, J. Polchinski and R. Roiban, *"Hidden symmetries of the* $\mathrm{AdS}_5 \times S^5$ *superstring"*, Phys. Rev. D69, 046002 (2004), hep-th/0305116.

[12] S. Novikov, *"The Hamiltonian formalism and a many valued analog of Morse theory"*, Usp.Mat.Nauk 37N5, 3 (1982).

[13] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, *"Classical/quantum integrability in AdS/CFT"*, JHEP 5, 24 (2004), hep-th/0402207. • N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, *"The algebraic curve of classical superstrings on* $\mathrm{AdS}_5 \times S^5$*"*, Commun. Math. Phys. 263, 659 (2006), hep-th/0502226.

[14] B. Vicedo, *"The method of finite-gap integration in classical and semi-classical string theory"*, J. Phys. A44, 124002 (2011), arxiv:0810.3402.

[15] H. Bethe, *"On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain"*, Z. Phys. 71, 205 (1931).

[16] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *"Quantum Inverse Scattering Method and Correlation Functions"*, Cambridge University Press (1997). • L. D. Faddeev, *"How Algebraic Bethe Ansatz works for integrable model"*, hep-th/9605187.

[17] J. A. Minahan and K. Zarembo, *"The Bethe-ansatz for* $\mathcal{N} = 4$ *super Yang-Mills"*, JHEP 0303, 013 (2003), hep-th/0212208.

[18] L. N. Lipatov, *"High-energy asymptotics of multicolor QCD and exactly solvable lattice*

models", hep-th/9311037. • L. D. Faddeev and G. P. Korchemsky, *"High-energy QCD as a completely integrable model"*, Phys. Lett. B342, 311 (1995), hep-th/9404173.

[19] N. Beisert, *"The complete one-loop dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory"*, Nucl. Phys. B676, 3 (2004), hep-th/0307015. • N. Beisert and M. Staudacher, *"The $\mathcal{N} = 4$ SYM integrable super spin chain"*, Nucl. Phys. B670, 439 (2003), hep-th/0307042.

[20] M. Luscher, *"Quantum Nonlocal Charges and Absence of Particle Production in the Two-Dimensional Nonlinear Sigma Model"*, Nucl. Phys. B135, 1 (1978). • D. Bernard, *"Hidden Yangians in 2-D massive current algebras"*, Commun. Math. Phys. 137, 191 (1991).

[21] L. Dolan, C. R. Nappi and E. Witten, *"A relation between approaches to integrability in superconformal Yang-Mills theory"*, JHEP 0310, 017 (2003), hep-th/0308089.

[22] N. Beisert, *"The* su(2|3) *dynamic spin chain"*, Nucl. Phys. B682, 487 (2004), hep-th/0310252. • N. Beisert, C. Kristjansen and M. Staudacher, *"The dilatation operator of conformal $\mathcal{N} = 4$ super Yang-Mills theory"*, Nucl. Phys. B664, 131 (2003), hep-th/0303060. • N. Beisert, V. Dippel and M. Staudacher, *"A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang-Mills"*, JHEP 0407, 075 (2004), hep-th/0405001.

[23] N. Beisert and M. Staudacher, *"Long-range* $\mathrm{PSU}(2,2|4)$ *Bethe ansaetze for gauge theory and strings"*, Nucl. Phys. B727, 1 (2005), hep-th/0504190.

[24] N. Beisert, *"The* su(2|2) *dynamic S-matrix"*, Adv.Theor.Math.Phys. 12, 945 (2008), hep-th/0511082. • N. Beisert, B. Eden and M. Staudacher, *"Transcendentality and crossing"*, J.Stat.Mech. 0701, P01021 (2007), hep-th/0610251.

[25] G. Arutyunov, S. Frolov and M. Staudacher, *"Bethe ansatz for quantum strings"*, JHEP 0410, 016 (2004), hep-th/0406256.

[26] R. Hernández and E. López, *"Quantum corrections to the string Bethe ansatz"*, JHEP 0607, 004 (2006), hep-th/0603204. • R. A. Janik, *"The* $\mathrm{AdS}_5 \times S^5$ *superstring worldsheet S-matrix and crossing symmetry"*, Phys. Rev. D73, 086006 (2006), hep-th/0603038. • N. Beisert, R. Hernández and E. López, *"A crossing-symmetric phase for* $\mathrm{AdS}_5 \times S^5$*"*, JHEP 0611, 070 (2006), hep-th/0609044.

[27] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, *"The off-shell symmetry algebra of the light-cone* $\mathrm{AdS}_5 \times S^5$ *superstring"*, J. Phys. A40, 3583 (2007), hep-th/0609157. • G. Arutyunov, S. Frolov and M. Zamaklar, *"The Zamolodchikov-Faddeev algebra for* $\mathrm{AdS}_5 \times S^5$ *superstring"*, JHEP 0704, 002 (2007), hep-th/0612229.

[28] M. Lüscher, *"Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States"*, Commun. Math. Phys. 104, 177 (1986). • M. Lüscher, *"Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States"*, Commun. Math. Phys. 105, 153 (1986). • Z. Bajnok and R. A. Janik, *"Four-loop perturbative Konishi from strings and finite size effects for multiparticle states"*, Nucl. Phys. B807, 625 (2009), arxiv:0807.0399.

[29] J. Ambjørn, R. A. Janik and C. Kristjansen, *"Wrapping interactions and a new source of corrections to the spin-chain/string duality"*, Nucl. Phys. B736, 288 (2006), hep-th/0510171.

[30] C. Sieg and A. Torrielli, *"Wrapping interactions and the genus expansion of the 2-point function of composite operators"*, Nucl. Phys. B723, 3 (2005), hep-th/0505071.

[31] A. B. Zamolodchikov, *"Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models"*, Nucl. Phys. B342, 695 (1990). • P. Dorey and R. Tateo, *"Excited states by analytic continuation of TBA equations"*, Nucl. Phys. B482, 639 (1996), hep-th/9607167.

[32] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable quantum field theories in finite volume: Excited state energies"*, Nucl. Phys. B489, 487 (1997), hep-th/9607099.

[33] C. Yang and C. Yang, *"Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction"*, J.Math.Phys. 10, 1115 (1969).

[34] G. Arutyunov and S. Frolov, *"String hypothesis for the* $\mathrm{AdS}_5 \times S^5$ *mirror"*, JHEP 0903, 152 (2009), arxiv:0901.1417. • G. Arutyunov and S. Frolov, *"Thermodynamic Bethe Ansatz for the* $\mathrm{AdS}_5 \times S^5$ *Mirror Model"*, JHEP 0905, 068 (2009), arxiv:0903.0141. • D. Bombardelli, D. Fioravanti and R. Tateo, *"Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal"*, J. Phys. A42, 375401 (2009), arxiv:0902.3930. • N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *"Integrability for the Full Spectrum of Planar AdS/CFT II"*, Lett. Math. Phys. 91, 265 (2010), arxiv:0902.4458.

[35] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *"Quantum spectral curve for* $\mathrm{AdS}_5/\mathrm{CFT}_4$*"*, Phys.Rev.Lett. 112, 011602 (2014), arxiv:1305.1939.

[36] R. J. Baxter, *"Partition function of the eight vertex lattice model"*, Annals Phys. 70, 193 (1972), [Annals Phys.281,187(2000)].

[37] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz"*, Commun. Math. Phys. 177, 381 (1996), hep-th/9412229.

[38] R. Benichou, *"First-principles derivation of the AdS/CFT Y-systems"*, JHEP 1110, 112 (2011), arxiv:1108.4927. • F. Delduc, M. Magro and B. Vicedo, *"Alleviating the non-ultralocality of the* AdS_5 *x* S^5 *superstring"*, JHEP 1210, 061 (2012), arxiv:1206.6050.

[39] A. Torrielli, *"Lectures on Classical Integrability"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[40] F. Loebbert, *"Lectures on Yangian Symmetry"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[41] D. Bombardelli, *"Lectures on S-Matrices and Integrability"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[42] F. Levkovich-Maslyuk, *"Lectures on the Bethe Ansatz"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[43] S. J. van Tongeren, *"Introduction to the Thermodynamic Bethe Ansatz"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[44] S. Negro, *"Lectures on Integrable Structures in Quantum Field Theory and Massive ODE/IM Correspondence"*, arxiv:1606.xxxxx, in: *"An integrability primer for the gauge-gravity correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[45] O. Babelon, D. Bernard and M. Talon, *"Introduction to Classical Integrable Systems"*, Cambridge University Press (2003), Cambridge Books Online. • L. D. Faddeev and L. A. Takhtajan, *"Hamiltonian Methods in the Theory of Solitons"*, Berlin, Germany: Springer 592 p. (Springer Series in Soviet Mathematics) (1987). • S. Novikov, S. V. Manakov, L. P. Pitaevsky and V. E. Zakharov, *"Theory of Solitons. The Inverse Scattering Method."*, New York, USA: Consultants Bureau 276 p. (Contemporary Soviet Mathematics) (1984).

[46] A. Belavin and V. Drinfeld, *"Solutions of the Classical Yang–Baxter Equation for Simple Lie Algebras"*, Funct. Anal. Appl. 16, 159 (1982).

[47] A. A. Tseytlin, *"Review of AdS/CFT Integrability, Chapter II.1: Classical* $\mathrm{AdS}_5 \times S^5$ *string solutions"*, Lett.Math.Phys. 99, 103 (2010), arxiv:1012.3986. • M. Magro, *"Review of AdS/CFT Integrability, Chapter II.3: Sigma Model, Gauge Fixing"*, Lett.Math.Phys. 99, 149 (2010), arxiv:1012.3988.

[48] D. Bernard, *"An Introduction to Yangian Symmetries"*, Int. J. Mod. Phys. B7, 3517 (1993), hep-th/9211133. • N. J. MacKay, *"Introduction to Yangian symmetry in integrable field theory"*, Int. J. Mod. Phys. A20, 7189 (2005), hep-th/0409183.

[49] N. Beisert, *"On Yangian Symmetry in Planar N=4 SYM"*, arxiv:1004.5423. • T. Bargheer, N. Beisert and F. Loebbert, *"Exact Superconformal and Yangian Symmetry of Scattering Amplitudes"*, J. Phys. A44, 454012 (2011), arxiv:1104.0700. • A. Torrielli, *"Yangians, S-matrices and* AdS/CFT*"*, J.Phys. A44, 263001 (2011), arxiv:1104.2474. • F. Spill, *"Yangians in Integrable Field Theories, Spin Chains and Gauge-String Dualities"*, Rev.Math.Phys. 24, 1230001 (2012), arxiv:1201.1884.

[50] M. Gaudin, *"Thermodynamics of the Heisenberg Ising ring for* $\Delta \geq 1$*"*, Phys.Rev.Lett. 26, 1301 (1971). • M. Takahashi and M. Suzuki, *"A reply to the comments of Johnson et al. on the thermodynamics of the Heisenberg-Ising ring for* $|\Delta| < 1$*"*, Phys. Lett. 41A, 81 (1972).

[51] L. F. Alday, D. Gaiotto and J. Maldacena, *"Thermodynamic Bubble Ansatz"*, JHEP 1109, 032 (2011), arxiv:0911.4708.

[52] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable structure of conformal field theory. 2. Q operator and DDV equation"*, Commun. Math. Phys. 190, 247 (1997), hep-th/9604044. • V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *"Integrable structure of conformal field theory. 3. The Yang-Baxter relation"*, Commun. Math. Phys. 200, 297 (1999), hep-th/9805008.

[53] V. V. Bazhanov, A. N. Hibberd and S. M. Khoroshkin, *"Integrable structure of W(3) conformal field theory, quantum Boussinesq theory and boundary affine Toda theory"*, Nucl. Phys. B622, 475 (2002), hep-th/0105177. • V. V. Bazhanov and S. L. Lukyanov, *"Integrable structure of Quantum Field Theory: Classical flat connections versus quantum stationary states"*, JHEP 1409, 147 (2014), arxiv:1310.4390.

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Lectures on Classical Integrability

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Abstract

We review some essential aspects of classically integrable systems. The detailed outline of the lectures consists of

1. Introduction and motivation, with historical remarks;
2. Liouville theorem and action-angle variables, with examples (harmonic oscillator, Kepler problem);
3. Algebraic tools: Lax pairs, monodromy and transfer matrices, classical r-matrices and exchange relations, non-ultralocal Poisson brackets, with examples (non-linear Schrödinger model, principal chiral field);
4. Features of classical r-matrices: Belavin-Drinfeld theorems, analyticity properties, and lift of the classical structures to quantum groups;
5. Classical inverse scattering method to solve integrable differential equations: soliton solutions, spectral properties and the Gel'fand-Levitan-Marchenko equation, with examples (KdV equation, Sine-Gordon model).

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Contents

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1. Introduction and Motivation

In this section, we give a very short introduction and motivation to the subject. It would be a titanic effort to provide even just an adequate list of references. Therefore we will simply mention a few, relying for the others on the many reviews and books available by now on this topic.

Let us point out that the main source for these lecture notes is the classic textbook by Babelon, Bernard and Talon [1]. The reader is also encouraged to consult [2–5].

1.1. Historical Remarks

Soon after the formulation of Newton's equations, people have tried to find exact solutions for interesting non-trivial cases. The Kepler problem was exactly solved by Newton himself. Nevertheless, apart from that, only a handful of other problems could be treated exactly.

In the 1800s Liouville refined the notion of *integrability* for Hamiltonian systems, providing a general framework for solving particular dynamical systems *by quadratures*. However, it was not until the 1900s that a more or less systematic method was developed. This goes under the name of the *classical inverse scattering method*. It was invented by Gardner,

Green, Kruskal and Miura in 1967, when they successfully applied it to solve the Korteweg-deVries (KdV) equation of fluid mechanics [6], and it was further developed in [7, 8].

The quantum mechanical version of the inverse scattering method was then elaborated during the following decade by the Leningrad – St. Petersburg school (see for instance [9]), with Ludwig Faddeev as a head, and among many others Korepin, Kulish, Reshetikhin, Sklyanin, Semenov Tian-Shansky and Takhtajan. They established a systematic approach to integrable[1] quantum mechanical systems which makes connection to Drinfeld and Jimbo's theory of quantum groups [11], paving the way to the algebraic reformulation of the problem. This approach has the power of unifying in a single mathematical framework integrable quantum field theories (cf. [12]) together with lattice spin systems (cf. [13]).

It is probably appropriate to mention that *integrability* is still not the same as *solvability*. The fact that the two things very often go together is certainly what makes integrable theories so appealing, nevertheless one requires a distinction. There exist integrable systems which one cannot really solve to the very end, as well as exactly solvable systems which are not integrable[2]. *Solvability* ultimately depends on one's ability and computational power. *Integrability* rather refers to the property of a system to exhibit regular (quasi-periodic) *vs.* chaotic behaviour, and to its conservation laws. This enormously facilitates, and, most importantly, provides *general mathematical methods* for the exact solution of the problem. The same holds for the quantum mechanical version, where integrability implies very specific properties of the scattering theory and of the spectrum, as it will be amply illustrated during this school.

Nowadays, integrability appears in many different contexts within mathematics and mathematical physics. The list below includes only a very small subset of all the relevant research areas.

1. Classical integrability[3]

   Theory of PDEs, Differential Geometry, General Relativity, Fluid Mechanics.

2. Quantum integrability:

   Algebra, Knot Theory, Condensed Matter Physics, String Theory.

[1] The term *integrable* referring to a field-theoretical (infinite-dimensional) model originates from [10].

[2] The first statement is particularly relevant for quantum integrable systems on compact domains, such as those described in Stijn van Tongeren's lectures at this school [14]. For such systems, the spectrum is encoded in a set of typically quite complicated integral equations - *Thermodynamic Bethe Ansatz* - which can often only be solved via numerical methods. The second statement can instead be exemplified by those *dissipative* mechanical systems - such as a free-falling particle subject to air resistance - which are not integrable (and, in fact, not *conservative* either) but admit an exact solution. A more elaborated example is however the one of *exactly solvable chaotic systems* - see for instance [15].

[3] We would like to point out a conjecture put forward by Ward in 1985 [16] (see also [17]): *"... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge [Yang-Mills, ndr.] field equations (or its generalisations) by [4d dimensional, ndr.] reduction."* The vast freedom in the Lax pair formulation of the self-dual Yang-Mills equations is due to the arbitrariness of the gauge group. We thank M. Wolf for the information.

It would be impossible to do justice to all the applications of integrability in mathematics and physics; that is why we will just end this introduction with an *Ipse dixit.*

1.2. Why integrability?

L. Faddeev once wrote [18]

> *"One can ask, what is good in* $1+1$ *models, when our spacetime is* $3+1$ *dimensional. There are several particular answers to this question.*
>
> *(a) The toy models in* $1+1$ *dimension can teach us about the realistic field-theoretical models in a nonperturbative way. Indeed such phenomena as renormalisation, asymptotic freedom, dimensional transmutation (i.e. the appearance of mass via the regularisation parameters) hold in integrable models and can be described exactly.*
>
> *(b) There are numerous physical applications of the* $1+1$ *dimensional models in the condensed matter physics.*
>
> *(c) The formalism of integrable models showed several times to be useful in the modern string theory, in which the world-sheet is* 2 *dimensional anyhow. In particular the conformal field theory models are special massless limits of integrable models.*
>
> *(d) The theory of integrable models teaches us about new phenomena, which were not appreciated in the previous developments of Quantum Field Theory, especially in connection with the mass spectrum.*
>
> *(e) I cannot help mentioning that working with the integrable models is a delightful pastime. They proved also to be very successful tool for the educational purposes.*
>
> *These reasons were sufficient for me to continue to work in this domain for the last 25 years (including 10 years of classical solitonic systems) and teach quite a few followers, often referred to as Leningrad - St. Petersburg school."*

2. Integrability in Classical Hamiltonian Systems

In this section we review the notion of integrability for a Hamiltonian dynamical system, and how this can be used to solve the equations of motion.

2.1. Liouville Theorem

Let us take a Hamiltonian dynamical system with a $2d$-dimensional phase space M parameterised by the canonical variables

$$(q_\mu, p_\mu), \qquad \mu = 1, ..., d. \tag{2.1}$$

Let the Hamiltonian function be $H(q_\mu, p_\mu)$, where q_μ, p_μ denotes the collection of variables (2.1). We also require the Poisson brackets to be

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0, \qquad \{q_\mu, p_\nu\} = \delta_{\mu\nu}, \qquad \forall\ \mu, \nu = 1, ..., d. \tag{2.2}$$

One calls the system *Liouville integrable* if one can find d independent conserved quantities F_μ, $\mu = 1, ..., d$, in involution, namely

$$\{F_\mu, F_\nu\} = 0, \qquad \forall\ \mu, \nu = 1, ..., d. \tag{2.3}$$

Independence here refers to the linear independence of the set of one-forms dF_μ. Note that, since d is the maximal number of such quantities, and since conservation of all the F_μ means $\{H, F_\mu\} = 0\ \forall\, \mu = 1, ..., d$, then one concludes that

$$H = H(F_\mu), \tag{2.4}$$

i.e. the Hamiltonian itself is a function of the quantities F_μ.

Theorem (Liouville). *The equations of motion of a Liouville-integrable system can be solved "by quadratures"* [4].

Proof. Let us take the canonical one-form

$$\alpha \equiv \sum_{\mu=1}^{d} p_\mu\, dq_\mu \tag{2.5}$$

and consider the d-dimensional level submanifold

$$M_f \equiv \{(q_\mu, p_\mu) \in M | F_\mu = f_\mu\} \tag{2.6}$$

for some constants f_μ, $\mu = 1, ..., d$. Construct the function

$$S \equiv \int_C \alpha\ = \int_{q_0}^{q} \sum_{\mu=1}^{d} p_\mu\, dq_\mu, \tag{2.7}$$

where the open (smooth, non self-intersecting) path C lies entirely in M_f. In (2.7), one thinks of the momenta p_μ as functions of F_μ, which are constant on M_f, and of the coordinates (see comments in section 2.2).

The function S is well-defined as a function of the upper extremum of integration q (with q_0 thought as a convenient reference point), because the integral in its definition (2.7) does not depend on the path. This is a consequence (via Stokes' theorem) of the fact that $d\alpha = 0$ on M_f. We will now prove that $d\alpha = 0$ on M_f.

Proof. One has that

$$\omega \equiv d\alpha = \sum_{\mu=1}^{d} dp_\mu \wedge dq_\mu \tag{2.8}$$

is the symplectic form on M. Let us denote by X_μ the Hamiltonian vector field associated to F_μ, acting like

$$X_\mu(g) \equiv \{F_\mu, g\} = \sum_{\nu=1}^{d} \left(\frac{\partial F_\mu}{\partial q_\nu} \frac{\partial g}{\partial p_\nu} - \frac{\partial F_\mu}{\partial p_\nu} \frac{\partial g}{\partial q_\nu} \right) \tag{2.9}$$

[4] We may consider this as a synonym of *"by straightforward integration"*.

on any function g on M. Equivalently,

$$dF_\mu(.) = \omega(X_\mu, \cdot) \tag{2.10}$$

on vectors of the tangent space to M. Then, one has

$$X_\mu(F_\nu) = \{F_\mu, F_\nu\} = 0 \qquad \forall\ \mu, \nu = 1, ..., d \tag{2.11}$$

because of (2.3).

Eq. (2.11) then implies that the X_μ are tangent to the level manifold M_f and form a basis for the tangent space to M_f, since the F_μ are all independent. One therefore obtains that

$$\omega(X_\mu, X_\nu) = dF_\mu(X_\nu) = X_\nu(F_\mu) = 0, \tag{2.12}$$

hence $\omega = 0$ on M_f. □

At this point, we simply regard S as a function of F_μ and of the upper integration point q_μ, and conclude that

$$dS \ = \sum_{\nu=1}^{d} p_\nu \, dq_\nu \ + \sum_{\nu=1}^{d} \frac{\partial S}{\partial F_\nu} \, dF_\nu \equiv \sum_{\nu=1}^{d} p_\nu \, dq_\nu \ + \sum_{\nu=1}^{d} \psi_\nu \, dF_\nu, \tag{2.13}$$

where we have defined

$$\psi_\mu \equiv \frac{\partial S}{\partial F_\mu}. \tag{2.14}$$

From $d^2 S = 0$ we deduce

$$\omega = \sum_{\mu=1}^{d} dF_\mu \ \wedge \ d\psi_\mu \tag{2.15}$$

which shows that the transformation $(q_\mu, p_\mu) \ \to \ (\psi_\mu, F_\mu)$ is canonical. Moreover, all the new momenta F_μ are constants of motion, hence the time-evolution of the new coordinates is simply

$$\frac{d\psi_\mu}{dt} \ = \ \frac{\partial H}{\partial F_\mu} \ = \text{ constant in time} \tag{2.16}$$

due to (2.4) and conservation of the F_μ. The evolution of the new coordinates is therefore linear in time, as can be obtained by straightforward integration of (2.16) (namely, performing one *quadrature*). □

2.2. Action-angle variables

The manifold M_f defined by the equations $F_\mu(q_\nu, p_\nu) = f_\mu$ typically displays non-trivial cycles, corresponding to a non-trivial topology, therefore the new coordinates ψ_μ are in principle multi-valued. For instance, the d-dimensional anisotropic harmonic oscillator admits d conserved quantites in involution:

$$F_\mu = \frac{1}{2}(p_\mu^2 + \omega_\mu^2 \, q_\mu^2) \tag{2.17}$$

(where we have set the mass equal to 1 for convenience). One can see from (2.17) that the expression for $p_\mu = p_\mu(q_\nu, F_\rho)$, which is needed to construct S in (2.7), allows for two independent choices of sign. The level manifold M_f is diffeomorphic to a d-dimensional torus[5].

We shall put ourselves in the situation where the manifold M_f has exactly d non-trivial cycles C_μ, $\mu = 1, ..., d$. The *action* variables are then defined by

$$I_\mu = \frac{1}{2\pi} \oint_{C_\mu} \alpha, \tag{2.18}$$

depending only on the F_ν. We can therefore regard S in (2.7) as

$$S = S(I_\mu, q_\nu). \tag{2.19}$$

If we now define

$$\theta_\mu \equiv \frac{\partial S}{\partial I_\mu} \tag{2.20}$$

we have

$$\oint_{C_\mu} d\theta_\nu = \frac{\partial}{\partial I_\nu} \oint_{C_\mu} dS = \frac{\partial}{\partial I_\nu} \oint_{C_\mu} \sum_{\rho=1}^{d} \frac{\partial S}{\partial q_\rho} dq_\rho = \frac{\partial}{\partial I_\nu} \oint_{C_\mu} \alpha, \tag{2.21}$$

where we have used that $dI_\mu = 0$ along the contour since the I_μ are constant on M_f, and that $\frac{\partial S}{\partial q_\mu} = p_\mu$. From the very definition (2.18) we then get

$$\oint_{C_\mu} d\theta_\nu = 2\pi\, \delta_{\mu\nu}, \tag{2.22}$$

displaying how every variable θ_μ changes of an amount of 2π along their corresponding cycle C_μ. This shows that the θ_μ are *angle* variables parameterising the d-dimensional torus.

Examples

- Let us consider a one-dimensional harmonic oscillator with $\omega = 1$, such that the Hamiltonian reads

  $$H = \frac{1}{2}(p^2 + q^2). \tag{2.23}$$

  There is one conserved quantity $H = F > 0$, with level manifold

  $$M_f = \{(q,p)\,|\,q^2 + p^2 = 2F \equiv R^2 = \text{constant}\} = \{q = R\cos\alpha,\, p = R\sin\alpha, R > 0, \alpha \in [0, 2\pi)\}.$$

  We can immediately see that M_f is a circle of radius $R = \sqrt{2F}$ in phase space. If we calculate S from (2.7) we obtain, for $q_0 = R$,

  $$S = -R^2 \int_0^\alpha d\alpha'\, \sin^2\alpha' = F(-\alpha + \sin\alpha\, \cos\alpha). \tag{2.24}$$

[5]Notice that this is not a peculiar situation, rather it is quite generic. Under the assumption of compactness and connectedness, it is actually guaranteed by the *Arnold-Liouville* theorem.

At this point, we need to re-express S as a function of F and q, therefore we need to distinguish between the two branches $p > 0$ and $p < 0$, which introduces a multi-valuedness into S:

$$\alpha = \arccos \frac{q}{R}, \qquad \text{if } \alpha \in [0, \pi], \qquad \alpha = 2\pi - \arccos \frac{q}{R}, \qquad \text{if } \alpha \in (\pi, 2\pi). \tag{2.25}$$

If we define a function

$$g(q, F) \equiv \frac{q}{2}\sqrt{2F - q^2} - F \arccos \frac{q}{\sqrt{2F}}, \qquad q \in \Big[-\sqrt{2F}, \sqrt{2F} \Big], \tag{2.26}$$

then

$$S = g(q, F), \qquad \text{if } p \geqslant 0, \qquad S = -2\pi F - g(q, F), \qquad \text{if } p < 0. \tag{2.27}$$

Let us now calculate the new coordinate $\psi = \frac{\partial S}{\partial F}$. We obtain

$$\psi = -\arccos \frac{q}{\sqrt{2F}}, \qquad \text{if } p \geqslant 0, \qquad \psi = -2\pi + \arccos \frac{q}{\sqrt{2F}}, \qquad \text{if } p < 0, \tag{2.28}$$

from which we conclude

$$\psi = -\alpha. \tag{2.29}$$

We can immediately verify that ψ depends linearly on time, because we know that the solution to the equations of motion is

$$q = \sqrt{2F}\cos(t + \phi) \tag{2.30}$$

for an initial phase ϕ. We also find from (2.18)

$$I = \frac{1}{2\pi} S_{\alpha=2\pi} \; = \; -F, \tag{2.31}$$

hence the angle variable is given by

$$\theta = \frac{\partial S}{\partial I} = -\frac{\partial S}{\partial F} = -\psi = \alpha. \tag{2.32}$$

The phase space is foliated by circles of constant energy, and the action-angle variables are the polar coordinates (see Figure 1).

- *The Kepler Problem*

  The Kepler problem can generally be formulated as the motion of a three-dimensional particle of mass m in a central potential $V(r) = \frac{\beta}{r}$.

  The dimension of the phase space is 6, therefore finding 3 conserved charges in involution will prove the integrability of the problem. The angular momentum is conserved due to rotational symmetry, hence J^2, J_z and H form a Poisson-commuting set. They are also independent, hence the system is integrable.

  To exhibit the appropriate change of variables, it is convenient to use polar coordinates (r, θ, ϕ), where θ is the polar angle $\theta \in [0, \pi]$. The conserved quantities are:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) + V(r), \qquad J^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}, \qquad J_z = p_\phi. \tag{2.33}$$

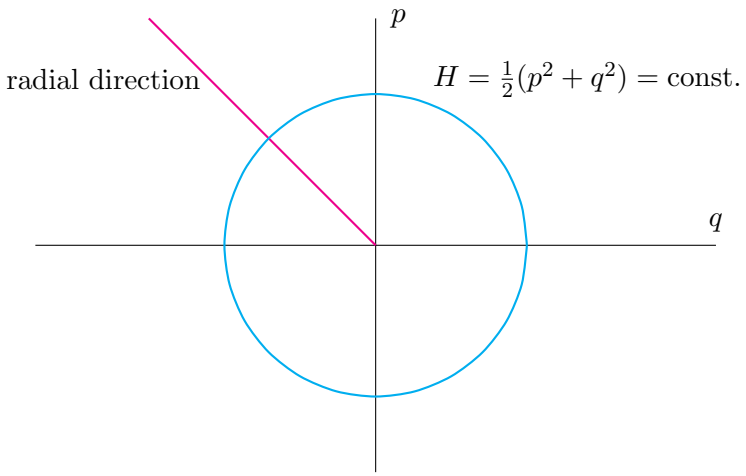

Figure 1: Foliation of phase space in circles of constant energy.

These relations can easily be inverted, and then plugged into the expression for the generating function S (2.7), with sign multi-valuedness. Noticing that, in this particular example, each polar momentum only depends on its conjugated polar coordinate, one obtains

$$S = \int^r dr \sqrt{2\Big(H - V(r)\Big) - \frac{J^2}{r^2}} + \int^\theta d\theta \sqrt{J^2 - \frac{J_z^2}{\sin^2\theta}} + \int^\phi d\phi\, J_z. \tag{2.34}$$

The new coordinate variables are

$$\psi_{J_z} = \frac{\partial S}{\partial J_z}, \qquad \psi_{J^2} = \frac{\partial S}{\partial J^2}, \qquad \psi_H = \frac{\partial S}{\partial H}. \tag{2.35}$$

As a consequence of (2.16), the first two coordinates in (2.35) are simply constant, while the third one obeys

$$t - t_0 = \psi_H = \int^r dr \frac{1}{\sqrt{2\Big(H - V(r)\Big) - \frac{J^2}{r^2}}}, \tag{2.36}$$

which is the standard formula for Keplerian motion. In order to complete the analysis, we should now define the action-angle variables and determine the frequencies of angular motion along the torus. Let us remark that this necessitates the explicit use of the torus cycles, as dictated by (2.18).

Notice that more independent quantities are conserved besides J^2, J_z and H, namely also J_x, J_y and the *Laplace-Runge-Lenz* vector:

$$\vec{A} = \vec{p} \times \vec{J} + m\,\beta\,\hat{r}, \tag{2.37}$$

where $\hat{r}$ is the unit vector in the radial direction. When the total number of independent conserved quantities (the d ones which are in involution plus the remaining ones) equals $d+m$, with $0 < m < d-1$, we will call the system *super-integrable*. When $m = d-1$, we call

the system *maximally super-integrable*. In the case of the Kepler problem, we have found 8 conserved quantities, namely $E, J^2, \vec{J}$ and $\vec{A}$. However, only five are independent, as we have three relations:

$$J^2 = \sum_{\mu=1}^{3} J_\mu^2, \qquad \vec{A}\cdot\vec{J} = 0, \qquad A^2 = m^2\beta^2 \,+\, 2mE\,J^2. \tag{2.38}$$

The Kepler problem is therefore maximally super-integrable, since $d = 3$.

Let us conclude this section by saying that, *locally* in phase space, one can reproduce much of the construction we have outlined for generic Hamiltonian systems, which might raise doubts about the peculiarity of integrability. The crucial distinction is that, for integrable systems, the procedure we have described extends *globally*. In particular, one has a global foliation of phase space by the M_f submanifolds, and a nice global geometric structure arising from this analysis.

3. Algebraic Methods

In this section we introduce the concepts of Lax pair, Monodromy and Transfer matrix, and Classical r-matrix. These quantities prominently feature in the so-called Inverse Scattering Method, which begins by recasting the Poisson brackets of the dynamical variables of a classically integrable system in a form which is most suitable for displaying the structure of its symmetries. At the end of this section we will briefly comment on the issue of *(non) ultra-locality* of the Lax-pair Poisson brackets.

3.1. Lax pairs

Suppose you can find two matrices L, M such that Hamilton's equations of motion can be recasted in the following form:

$$\frac{dL}{dt} \,=\, [M, L]. \tag{3.1}$$

The two matrices L, M are said to form a *Lax pair*. From (3.1), we can immediately obtain a set of conserved quantites:

$$O_n \,\equiv\, \mathrm{tr}\, L^n, \qquad \frac{dO_n}{dt} \,=\, \sum_{i=0}^{n-1} \mathrm{tr}\; L^i\,[M, L]L^{n-1-i} \,=\, 0, \qquad \forall\; n \text{ natural number}, \tag{3.2}$$

by simply opening up the commutator and using the cyclicity of the trace. Of course not all of these conserved charges are independent. Notice that (3.2) implies that the eigenvalues λ_α of L are conserved in time, since $O_n = \sum_\alpha \lambda_\alpha^n$.

Let us point out that the Lax pair is not unique[6], as there is at least a gauge freedom

$$L \longrightarrow g\,L\,g^{-1}, \qquad M \longrightarrow g\,M\,g^{-1} + \frac{dg}{dt}\,g^{-1}, \tag{3.3}$$

[6]For example, adding constant multiples of the identity to L and M preserves (3.1). There also exist particular models where one can describe the system using alternative Lax pairs of different ranks (cf. [19], pages 2-3).

with g an invertible matrix depending on the phase-space variables.

Example

- A Lax pair for the harmonic oscillator (with mass $m = 1$ in appropriate units) can be written down as follows:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} = p\,\sigma_3 + \omega q\,\sigma_1, \qquad M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix} = -i\frac{\omega}{2}\,\sigma_2. \tag{3.4}$$

  One can immediately check using (3.4) that the only non-zero components of (3.1) - those along σ_1 and σ_3 - are equivalent to Hamilton's equations of motion.

We will regard L and M as elements of some matrix algebra $\mathfrak{g}$, with the matrix entries being functions on phase space. For example, in the case of the harmonic oscillator (3.4) one can see that $\mathfrak{g}$ is the complexification of the $su(2)$ Lie algebra, which is isomorphic to $sl(2,\mathbb{C})$.

Even if we assume that we have found a Lax pair, and that we can obtain d independent conserved quantities, this does not yet guarantee their *involutivity*. Hence, we have not yet secured integrability. For that, we need an extra ingredient. We introduce the following notation:

$$X_1 \equiv X \otimes 1, \qquad X_2 \equiv 1 \otimes X \tag{3.5}$$

as elements of $\mathfrak{g} \otimes \mathfrak{g}$. Then, one has the following

Theorem. *The eigenvalues of L are in involution iff there exists an element $r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$, function of the phase-space variables, such that*

$$\{L_1, L_2\} \;=\; [r_{12}, L_1] - [r_{21}, L_2], \tag{3.6}$$

where $r_{21} = \Pi \circ r_{12}$, Π being the permutation operator acting on the two copies of $\mathfrak{g} \otimes \mathfrak{g}$.

The proof can be found in [1].

In order for the Jacobi identity to hold for the Poisson bracket (3.6) one needs to impose the following condition, defined on the triple tensor product $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$:

$$\begin{aligned} &[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] \;+ \\ &[L_2, [r_{13}, r_{21}] + [r_{23}, r_{21}] + [r_{23}, r_{31}] + \{L_3, r_{21}\} - \{L_1, r_{23}\}] \;+ \\ &[L_3, [r_{31}, r_{12}] + [r_{21}, r_{32}] + [r_{31}, r_{32}] + \{L_1, r_{32}\} - \{L_2, r_{31}\}] \;=\; 0. \end{aligned} \tag{3.7}$$

One can see from here that, if r_{12} is a constant independent of the dynamical variables, and if we furthermore require that

$$r_{12} = -r_{21}, \tag{3.8}$$

then we see that a *sufficient* condition for the Jacobi identity to be satisfied is

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \;=\; 0. \tag{3.9}$$

We call such an r a *constant classical r-matrix*, and (3.9) the *classical Yang-Baxter equation (CYBE)*.

Notice that another sufficient condition would be to have (3.9), but with a Casimir element instead of zero on the right hand side. This *modified Yang-Baxter equation* would lead us to a more general mathematical setting, which however goes beyond the scope of these lectures.

Examples

- The following matrix is a constant solution of the CYBE:

$$r = e \otimes h - h \otimes e, \qquad [h, e] = e. \tag{3.10}$$

  The algebra it is based upon is the triangular Borel subalgebra of $sl(2)$ generated by the Cartan element h and one of the roots, here denoted by e.

- The matrix r_{12} for the harmonic oscillator is non-constant (sometimes such r-matrices are called *dynamical* as opposed to the constant ones), and it reads

$$r_{12} = -\frac{\omega}{4H} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes L = -\frac{i\omega}{4H}\, \sigma_2 \otimes L, \tag{3.11}$$

  with $H = \frac{1}{2}(p^2 + \omega^2 q^2)$ being the energy, and $\{q, p\} = 1$ the canonical Poisson bracket. The eigenvalues of L in (3.4) are $\pm 2H$.

The most interesting case for our purposes will be when the Lax pair depends on an additional complex variable u, called the *spectral parameter*. This means that in some cases we can find a *family* of Lax pairs, parameterised by u, such that the equations of motion are equivalent to the condition (3.1) *for all values of* u. We will see that this fact has significant consequences for the inverse scattering method. Therefore, we are going to put ourselves in this situation from now on.

Example

- A Lax pair for the Kepler problem reads [20]

$$L = \frac{1}{2} \begin{pmatrix} -\chi_u[\vec{r}, \frac{d}{dt}\vec{r}] & \chi_u[\vec{r}, \vec{r}] \\ -\chi_u[\frac{d}{dt}\vec{r}, \frac{d}{dt}\vec{r}] & \chi_u[\vec{r}, \frac{d}{dt}\vec{r}] \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ M_0 & 0 \end{pmatrix}, \tag{3.12}$$

  where one has defined

$$\chi_u[\vec{a}, \vec{c}] \equiv \sum_{\mu=1}^{3} \frac{a_\mu c_\mu}{u - u_\mu}, \qquad M_0\, \vec{r} \equiv -\frac{1}{m}\, \nabla V(r) \tag{3.13}$$

  and $V(r) = \frac{\beta}{r}$. This Lax pair depends on three complex variables u_μ, $\mu = 1, 2, 3$, besides u which we take as a spectral parameter. Laurent-expanding Eq. (3.1) in u, one recovers the full set of Newton's equations $m \frac{d^2}{dt^2}\vec{r} = -\nabla V(r) = \beta \frac{\vec{r}}{r^3}$.

3.2. Field theories. Monodromy and Transfer matrices.

An important step we need to take is to generalise what we have reviewed so far for classical finite-dimensional dynamical systems, to the case of classical *field theories*. We will restrict ourselves to two-dimensional field theories, meaning one spatial dimension x and one time t. This means that we will now have equations of motion obtained from a classical field theory Lagrangian (Euler-Lagrange equations).

The notion of integrability we gave earlier, based on the Liouville theorem, is inadequate when the number of degrees of freedom becomes infinite, as it is the case for field theories. What we will do is to adopt the idea of a Lax pair, suitably modifying its definition, as a starting point to define an integrable field theory.

Suppose you can find two (spectral-parameter dependent) matrices L, M such that the Euler-Lagrange equations of motion can be recasted in the following form:

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = [M, L]. \tag{3.14}$$

We will call such field theories *classically integrable.*

The condition (3.14) is also the compatibility condition for the following *auxiliary linear problem*:

$$(\partial_x - L)\Psi = 0, \qquad (\partial_t - M)\Psi = 0, \tag{3.15}$$

as can be seen by applying ∂_t to the first equation, ∂_x to the second equation, subtracting the two and using (3.15) one more time. We will make use of the auxiliary linear problem later on, when we will discuss solitons.

The two matrices L, M in (3.14) are also said to form a *Lax pair*, and one can in principle obtain a sequence of conserved quantities for the field theory by following a well-defined procedure. Such a procedure works as follows.

Let us introduce the so-called *monodromy matrix*

$$T(u) = P \exp\left[\int_{s_-}^{s_+} L(x,t,u)dx\right] \tag{3.16}$$

where P denotes a path-ordering with greater x to the left, s_- and s_+ are two points on the spatial line, and u is the spectral parameter. This object can be thought of as implelementing a parallel transport along the segment $[s_-, s_+]$, in accordance with the fact that the Lax pair can be thought of as a connection.

If we differentiate $T(u)$ with respect to time, we get

$$\begin{aligned}
\partial_t T &= \int_{s_-}^{s_+} dx\, P \exp\left[\int_x^{s_+} L(x',t,u)dx'\right] \left[\partial_t L(x,t,u)\right] P \exp\left[\int_{s_-}^{x} L(x',t,u)dx'\right] \\
&= \int_{s_-}^{s_+} dx\, P \exp\left[\int_x^{s_+} L(x',t,u)dx'\right] \left(\frac{\partial M}{dx} - [L,M]\right) P \exp\left[\int_{s_-}^{x} L(x',t,u)dx'\right] \\
&= \int_{s_-}^{s_+} dx\, \partial_x \left(P \exp\left[\int_x^{s_+} L(x',t,u)dx'\right] M\ P \exp\left[\int_{s_-}^{x} L(x',t,u)dx'\right]\right) \\
&= M(s_+)\, T - T\, M(s_-),
\end{aligned} \tag{3.17}$$

where at some stage we have used (3.14). At this point, if we consider pushing s_- and s_+ towards the extrema $S_\pm$ of the spatial domain, respectively, and assume for definiteness this domain to be the compact segment $[0, 2\pi]$ with periodic boundary conditions on the fields, we obtain

$$\partial_t T \,=\, [M(0,t,u), T]. \tag{3.18}$$

This implies that the trace of T, called the *transfer matrix*

$$\mathfrak{t}(u) \,\equiv \operatorname{tr} T(u), \tag{3.19}$$

is conserved *for all* u. By expanding in u, one obtains a family of conserved charges, which are the coefficients of the expansion. For instance, if $\mathfrak{t}(u)$ is analytic near the origin, one Taylor-expands

$$\mathfrak{t}(u) = \sum_{n\geqslant 0} Q_n\, u^n, \qquad \partial_t Q_n = 0,\ \forall\, n \geqslant 0. \tag{3.20}$$

This forms the starting point for the construction of the integrable structure.

Examples

- The Non-linear Schrödinger (NLS) model[7] is the classical non-relativistic $1+1$ dimensional field theory with Hamiltonian

$$H = \int_{-\infty}^{\infty} dx \left(\left| \frac{\partial \psi}{\partial x} \right|^2 + \kappa |\psi|^4 \right) \tag{3.21}$$

for a complex field $\psi(x)$ with Poisson brackets

$$\{\psi(x), \psi^*(y)\} = \delta(x-y), \tag{3.22}$$

and a real coupling constant κ. The name *non-linear Schrödinger* model comes from the fact that the equations of motion look like

$$i\frac{\partial \psi}{\partial t} = \{H, \psi\} = \frac{\partial^2 \psi}{\partial x^2} + 2\kappa |\psi|^2 \psi, \tag{3.23}$$

namely they coincide with the Schrödinger equation for a non-linear potential depending on the square modulus of the wave function itself.

The Lax pair reads

$$L = \begin{pmatrix} -i\frac{u}{2} & i\kappa\psi^* \\ -i\psi & i\frac{u}{2} \end{pmatrix}, \qquad M = \begin{pmatrix} i\frac{u^2}{2} + i\kappa|\psi|^2 & \kappa\frac{\partial \psi^*}{\partial x} - i\kappa u \psi^* \\ \frac{\partial \psi}{\partial x} + iu\psi & -i\frac{u^2}{2} - i\kappa|\psi|^2 \end{pmatrix}, \tag{3.24}$$

depending on a complex spectral parameter u. One can write the monodromy matrix as

$$T(u) = \begin{pmatrix} a(u) & \kappa\, b^*(u^*) \\ b(u) & a^*(u^*) \end{pmatrix}, \tag{3.25}$$

[7] We will follow [9] in this example.

where $a(u)$ and $b(u)$ admit the following power-series representation:

$$a(u) = e^{-i\frac{u}{2}(s_+ - s_-)}\Bigg[1 + \sum_{n=1}^{\infty}\kappa^n \int_{s_+>\xi_n>\eta_n>\xi_{n-1}\ldots>\eta_1>s_-} d\xi_1\ldots d\xi_n\, d\eta_1\ldots d\eta_n \\ e^{iu(\xi_1+\ldots+\xi_n-\eta_1-\ldots-\eta_n)}\psi^*(\xi_1)\ldots\psi^*(\xi_n)\,\psi(\eta_1)\ldots\psi(\eta_n)\Bigg],$$

$$b(u) = -i\, e^{i\frac{u}{2}(s_+ + s_-)}\Bigg[1 + \sum_{n=1}^{\infty}\kappa^n \int_{s_+>\eta_{n+1}>\xi_n>\eta_n>\xi_{n-1}\ldots>\eta_1>s_-} d\xi_1\ldots d\xi_n\, d\eta_1\ldots d\eta_{n+1} \\ e^{iu(\xi_1+\ldots+\xi_n-\eta_1-\ldots-\eta_{n+1})}\psi^*(\xi_1)\ldots\psi^*(\xi_n)\,\psi(\eta_1)\ldots\psi(\eta_{n+1})\Bigg].$$

Some of the conserved charges one obtains by an appropriate expansion[8], and in the limit of infinite domain $s_\pm \to \pm\infty$, read

$$\mathfrak{I}_1 = \int_{-\infty}^{\infty} dx|\psi|^2, \qquad \mathfrak{I}_2 = \frac{i}{2}\int_{-\infty}^{\infty} dx\left(\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right), \qquad \mathfrak{I}_3 = H, \\ \mathfrak{I}_4 = i\int_{-\infty}^{\infty} dx\left(\frac{\partial^3\psi^*}{\partial x^3}\psi - 3\frac{\partial\psi^*}{\partial x}\psi|\psi|^2\right), \qquad \text{etc.} \tag{3.27}$$

Upon quantisation, the first charge corresponds to the particle number, the second one to the momentum, the third one to the Hamiltonian.

Action-angle type variables[9] for the NLS can be obtained in the following fashion. If we define

$$\varphi(u) = \frac{b(u)}{|b(u)|}\sqrt{\frac{\log a(u)}{\kappa\pi}}, \tag{3.28}$$

then this new variable is such that

$$\{\varphi(u), \varphi^*(u')\} = i\delta(u-u'), \qquad \frac{d}{dt}\varphi(u) = -iu\,\varphi(u). \tag{3.29}$$

The second equation guarantees that φ is the exponential of an angle variable.

In the new variables, the infinite tower of conserved charges - cf. (3.27) - collectively read:

$$\mathfrak{I}_m = \int_{-\infty}^{\infty} d\mu\,\mu^{m-1}|\varphi(\mu)|^2, \qquad m = 1, 2, \ldots \tag{3.30}$$

[8] In this example one actually Laurent-expands

$$\log a(u) = i\kappa\sum_{m=1}^{\infty}\mathfrak{I}_m u^{-m}. \tag{3.26}$$

The justification for this is that $a(u)$ itself commutes with the Hamiltonian, and can be proven to be an equally good generating function for the conserved charges [21, 22]. Moreover, one can show that $\{a(u), a(u')\} = 0$, hence the charges generated by (3.26) are all in involution with each other. This will be made more systematic in the following sections, where it will be seen to follow from the *Sklyanin exchange relations.*

[9] Let us quote Sklyanin's original words [9]: *"The concept of "action-angle variables" we shall treat here broadly, calling such any canonical variables in which the Hamiltonian H can be written as a quadratic form (and the equations of motion, correspondingly, become linear)."*

- The Sine-Gordon equation for a real scalar field ϕ in $1+1$ space-time dimensions reads

$$\partial_t^2\phi \,-\, \partial_x^2\phi \,=\, -\,\frac{8m^2}{\beta}\,\sin(2\beta\phi), \tag{3.31}$$

for m and β two constants. One can recast this equation as a Lax pair condition, with a Lax pair given by

$$\begin{aligned} L &= i\begin{pmatrix} \frac{\beta}{2}\partial_t\phi & mu\,e^{i\beta\phi} - \frac{m}{u}e^{-i\beta\phi} \\ mu\,e^{-i\beta\phi} - \frac{m}{u}e^{i\beta\phi} & -\frac{\beta}{2}\partial_t\phi \end{pmatrix}, \\ M &= i\begin{pmatrix} \frac{\beta}{2}\partial_x\phi & -mu\,e^{i\beta\phi} - \frac{m}{u}e^{-i\beta\phi} \\ -mu\,e^{-i\beta\phi} - \frac{m}{u}e^{i\beta\phi} & -\frac{\beta}{2}\partial_x\phi \end{pmatrix}, \end{aligned} \tag{3.32}$$

depending on a spectral parameter u.

3.3. Poisson structure and the problem of Non-Ultralocality

In the spirit of (3.6) we now consider the Poisson brackets between two Lax pair elements L, this time taken at two distinct positions x and y and for different spectral parameters u and u'. Suppose that the canonical Poisson brackets imposed on the fields have the following consequence for L:

$$\{L_1(x,t,u), L_2(y,t,u')\} \,=\, [r_{12}(u-u'), L_1(x,t,u) + L_2(y,t,u')]\,\delta(x-y), \tag{3.33}$$

with similar conventions as in (3.5). Let us also assume that the r-matrix $r_{12}(u-u')$ does not itself depend on the fields[10], and satisfies

$$r_{12}(u-u') = -r_{21}(u'-u). \tag{3.34}$$

Theorem (Sklyanin Exchange Relations). *Given (3.33), the Poisson brackets of the monodromy matrix satisfy*

$$\{T_1(u), T_2(u')\} \,=\, [r_{12}(u-u'), T_1(u)T_2(u')]. \tag{3.35}$$

From this, one can immediately conclude that the conserved charges generated by the transfer matrix are all in involution. Indeed, tracing by $\mathrm{tr}_1 \otimes \mathrm{tr}_2$ both sides of (3.35), one obtains

$$\{\mathfrak{t}(u), \mathfrak{t}(u')\} = 0, \tag{3.36}$$

where we have used cyclicity of $\mathrm{tr}_1 \otimes \mathrm{tr}_2$ which is the natural trace operation on $\mathfrak{g}\otimes\mathfrak{g}$. By expanding (3.36) we obtain the desired involution property of the charges. For analytic functions,

$$\mathfrak{t}(u) = \sum_{n\geqslant 0} Q_n\, u^n, \qquad \{Q_n, Q_m\} = 0 \quad \forall\; m,n \geqslant 0. \tag{3.37}$$

The variables L and T can therefore be thought of as the most convenient variables to display the integrable structure of the model. It will not come as a surprise then that quantisation best

[10] In principle, we shall not dub this r-matrix as *constant* any longer, because of the dependence on the spectral parameter.

proceeds from the Sklyanin relations, in what constitutes the backbone of the Quantum Inverse Scattering Method (QISM) [9].

With the assumptions described after (3.33), the Jacobi identity for Sklyanin's exchange relations admits again as a sufficient condition the *classical Yang-Baxter equation* with spectral parameter, namely

$$[r_{12}(u_1-u_2),r_{13}(u_1-u_3)]+[r_{12}(u_1-u_2),r_{23}(u_2-u_3)]+[r_{13}(u_1-u_3),r_{23}(u_2-u_3)] \;=\; 0. \tag{3.38}$$

The Poisson brackets (3.33) are called *ultra-local*, because they only display the Dirac delta function and not its derivatives. Whenever higher derivatives of $\delta(x-y)$ are present, one speaks of *non ultra-local* Poisson structures. In the latter case, one cannot obtain a formula like (3.35), and quantisation does not proceed along the standard lines of the QISM.

Maillet brackets

There is a situation where a significant amount of progress has been made, despite the presence of non ultra-locality. This is when the Poisson brackets between the spatial component of the Lax pair assume the form

$$\begin{aligned}\{L_1(x,t,u),L_2(y,t,u')\} \;=\; &\delta(x-y)\,[r_-(u,u'),L_1(x,t,u)] \\ &+\delta(x-y)\,[r_+(u,u'),L_2(y,t,u')] \;+\; \delta'(x-y)\,\Big(r_-(u,u')-r_+(u,u')\Big),\end{aligned} \tag{3.39}$$

for a choice of an (r,s)-matrix pair satisfying a mixed Yang-Baxter type equation:

$$r_+ \;=\; r+s, \qquad r_- \;=\; r-s, \tag{3.40}$$

$$\begin{aligned}&[(r+s)_{13}(u_1,u_3),(r-s)_{12}(u_1,u_2)] \;+\; [(r+s)_{23}(u_2,u_3),(r+s)_{12}(u_1,u_2)] \\ &\qquad +[(r+s)_{23}(u_2,u_3),(r+s)_{13}(u_1,u_3)] \;=\; 0\end{aligned} \tag{3.41}$$

(this is again to ensure the Jacobi identity of the brackets). The Poisson brackets for the (classical) monodromy matrix $T \equiv P\exp\int L$ have been derived from (3.39) by careful treatment of the ambiguity arising from the non ultra-locality [23, 24], and read[11]

$$\begin{aligned}\{T(u)\otimes 1, 1\otimes T(u')\} \;=\; &[r(u,u'),T(u)\otimes T(u')] \\ &-[1\otimes T(u')]\,s(u,u')\,[T(u)\otimes 1] \;+\; [T(u)\otimes 1]\,s(u,u')[1\otimes T(u')].\end{aligned} \tag{3.42}$$

By taking the trace of (3.42), one can still show that an infinite set of classically conserved charges in involution is generated by $\operatorname{tr} T(u)$. The problem is that no quantisation procedure has been so

[11] It is important to remark that the brackets (3.42) do not satisfy the Jacobi identity. This violation is connected to a naive equal-point limiting procedure, occurring when one first evaluates $\{T(u)\otimes 1, 1\otimes T(u')\}$ for different values of $s_\pm$ in each of the two factors. A careful treatment of this singularity is provided in [24], where a way to restore the Jacobi identity is defined via a more elaborated symmetric-limit procedure.

far fully established for the brackets (3.42). The quantum S-matrix can nevertheless be fixed by symmetries in most of the interesting cases.

Example

- The Principal Chiral Model provides a standard example of Maillet structure [23, 25]. This is the theory of an element g of a compact group G, with Lagrangian

$$\mathcal{L} = -\frac{1}{2\gamma}\,\mathrm{tr}\, j_\mu\, j^\mu, \qquad j_\mu = (\partial_\mu g) g^{-1}$$

admitting (left,right) global symmetry $g \to \left(e^{if} g,\, g\, e^{if}\right)$. In Minkowski signature, $x^0 = t$, $x^1 = x$, and conservation of j reads $\partial_\mu j^\mu = \partial_0 j_0 - \partial_1 j_1 = 0$. The constant γ is the coupling of the theory. Both f and the (left,right) currents

$$j^L_\mu = j_\mu = (\partial_\mu g) g^{-1}, \qquad j^R_\mu = -g^{-1}\,(\partial_\mu g)$$

belong to the Lie algebra of G. The currents are flat (cf. *Maurer-Cartan* one-forms), *i.e.*

$$\partial_\mu j^L_\nu - \partial_\nu j^L_\mu - [j^L_\mu, j^L_\nu] = 0, \qquad \partial_\mu j^R_\nu - \partial_\nu j^R_\mu - [j^R_\mu, j^R_\nu] = 0. \tag{3.43}$$

The Lax pair reads

$$L = \frac{u\, j_0 \,+\, j_1}{1-u^2}, \qquad M = \frac{u\, j_1 \,+\, j_0}{1-u^2}. \tag{3.44}$$

The (r, s) pair reads

$$r(u,u') \,=\, \frac{1}{2}\,\frac{\zeta(u)+\zeta(u')}{u-u'}\,\mathcal{C}_\otimes \qquad s(u,u') \,=\, \frac{1}{2}\,\frac{\zeta(u)-\zeta(u')}{u-u'}\,\mathcal{C}_\otimes, \tag{3.45}$$

with

$$\zeta(u) = \gamma\frac{u^2}{u^2-1}, \qquad \mathcal{C}_\otimes = \sum_{a,b} \kappa_{ab}\; t_a \otimes t_b, \tag{3.46}$$

in terms of the Lie algebra generators t^a. Indices in (3.46) are saturated with the Killing form κ_{ab} (see footnote 14 in the following).

We would like to conclude this section by mentioning that the Lax-pair formalism allows to derive a set of relations, called *finite-gap equations*, which are the first step to a semi-classical analysis of the spectrum of the integrable system. This is treated in detail in Fedor Levkovich-Maslyuk's lectures at this school [26], where the finite-gap equations are derived by taking a semi-classical limit of the quantum Bethe ansatz equations.

4. Classical r-matrices

In this section, we discuss the properties of classical r-matrices, most notably their analytic structure and their relation to infinite-dimensional algebras. The highlights of this section will be the famous Belavin-Drinfeld theorems.

4.1. Belavin-Drinfeld theorems

Mathematicians have studied classical r-matrices and have classified them under specific assumptions. We begin by presenting the most important theorems in this area [27–29].

Theorem (Belavin Drinfeld I). *Let $\mathfrak{g}$ be a finite-dimensional simple*[12] *Lie algebra, and $r = r(u_1 - u_2) \in \mathfrak{g} \otimes \mathfrak{g}$ a solution of the (spectral-parameter dependent) classical Yang-Baxter equation (3.38). Furthermore, assume one of the following three equivalent conditions to hold:*

- *(i) r has at least one pole in the complex plane $u \equiv u_1 - u_2$, and there is no Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ such that $r \in \mathfrak{g}' \otimes \mathfrak{g}'$ for any u,*
- *(ii) $r(u)$ has a simple pole at the origin, with residue proportional to $\sum_a t_a \otimes t_a$, with t_a a basis in $\mathfrak{g}$ orthonormal with respect to a chosen nondegenerate invariant bilinear form*[13]*,*
- *(iii) the determinant of the matrix $r_{ab}(u)$ obtained from $r(u) = \sum_{ab} r_{ab}(u)\, t_a \otimes t_b$ does not vanish identically.*

Under these assumptions, $r_{12}(u) = -r_{21}(-u)$ where $r_{21}(u) = \Pi \circ r_{12}(u) = \sum_{ab} r_{ab}(u)\, t_b \otimes t_a$, and $r(u)$ can be extended meromorphically to the entire u-plane. All the poles of $r(u)$ are simple, and they form a lattice Γ. One has three possible equivalence classes of solutions: "elliptic" - when Γ is a two-dimensional lattice -, "trigonometric" - when Γ is a one-dimensional array -, or "rational"- when $\Gamma = \{0\}$-, respectively.

The assumption of difference-form is not too restrictive, thanks to the following theorem by the same authors [29]:

Theorem (Belavin Drinfeld II). *Assume the hypothesis of Belavin-Drinfeld I theorem but $r = r(u_1, u_2)$* **not** *to be of difference form, with the classical Yang-Baxter equation being the natural generalisation of (3.38):*

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] + [r_{12}(u_1, u_2), r_{23}(u_2, u_3)] + [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] \;=\; 0. \tag{4.1}$$

Now the three statements (i), (ii) and (iii) are not any longer immediately equivalent, and we will only retain (ii). Assume the dual Coxeter number[14] *of $\mathfrak{g}$ to be non vanishing. Then, there exists a transformation which reduces r to a difference form.*

Proof. Without loss of generality, we can assume that the r-matrix will behave as

$$r \sim \frac{\sum_a t_a \otimes t_a}{u_1 - u_2} + g(u_1, u_2) \tag{4.2}$$

[12] A Lie algebra is simple when it has no non-trivial ideals, or, equivalently, its only ideals are $\{0\}$ and the algebra itself. An *ideal* is a subalgebra such that the commutator of the whole algebra with the ideal is contained in the ideal.

[13] Such a residue can be identified with the quadratic Casimir $\mathcal{C}_\otimes$ in $\mathfrak{g} \otimes \mathfrak{g}$.

[14] The dual Coxeter number $\mathfrak{c}_2$ is defined as $\sum_{ab} f_{abc}\, f_{abd} = \mathfrak{c}_2\, \delta_{cd}$, and it is related to trace of the quadratic Casimir in the adjoint representation, *i.e.* $\sum_a [t_a, [t_a, x]] = \mathfrak{c}_2\, x$, $\forall\, x \in \mathfrak{g}$. The Killing form is nothing else but $\kappa_{cd} = \sum_{ab} f_{abc}\, f_{abd} = \mathfrak{c}_2\, \delta_{cd}$.

near the origin. If it does not, then

$$r \sim \frac{\xi(u_1)\sum_a t_a \otimes t_a}{u_1 - u_2} + g(u_1, u_2), \tag{4.3}$$

and the change of variables

$$u = u(v), \qquad u'(v) = \xi(u(v)) \tag{4.4}$$

will make the residue equal to 1. In fact, near $v_1 = v_2$, one has

$$\frac{\xi(u_1)}{u_1 - u_2} \sim \frac{\xi(u_1)}{u(v_2) + u'(v_2)(v_1 - v_2) - u(v_2)} \sim \frac{1}{v_1 - v_2} \tag{4.5}$$

due to (4.4). The function g can be taken to be holomorphic in a sufficiently small neighbourhood of the origin.

Expanding (4.1) near the point $u_2 = u_3$, we get

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_2)] + [r_{12}(u_1, u_2) + r_{13}(u_1, u_2), g_{23}(u_2, u_2)] + \left[\sum_{ab} r_{ab}(u_1, u_2) t_a \otimes \Delta(t_b), 1 \otimes \frac{\sum_c t_c \otimes t_c}{u_2 - u_3}\right] + \left[\partial_{u_2} r_{12}(u_1, u_2), 1 \otimes \sum_a t_a \otimes t_a\right] = 0$$

where $\Delta(t_a) = t_a \otimes 1 + 1 \otimes t_a$ coincides with the *trivial coproduct* on $\mathfrak{g}$, when $\mathfrak{g}$ is regarded as a *bialgebra* [30]. However, the third term cancels out because $\sum_a t_a \otimes t_a$ is the quadratic Casimir of $\mathfrak{g} \otimes \mathfrak{g}$, hence $[\Delta(t_b), \sum_a t_a \otimes t_a] = 0$.

Now we apply the commutator map $x \otimes y \to [x, y]$ to the spaces 2 and 3 in the above equation, and use the fact that the dual Coxeter number is non-zero (and equal to 1 if we use appropriate conventions). After using the Jacobi identity, we get

$$\sum_{abcd} r_{ab}(u_1, u_2) r_{cd}(u_1, u_2)[t_a, t_c] \otimes [t_b, t_d] + [r(u_1, u_2), 1 \otimes h(u_2)] + \partial_{u_2} r(u_1, u_2) = 0, \tag{4.6}$$

where $h(u) \equiv g_{ab}(u, u)[t_a, t_b]$.

We can repeat the same process on the variables u_1 and u_2, and the spaces 1 and 2. We obtain

$$\sum_{abcd} r_{ab}(u_1, u_3) r_{cd}(u_1, u_3)[t_a, t_c] \otimes [t_b, t_d] + [h(u_1) \otimes 1, r(u_1, u_3)] - \partial_{u_1} r(u_1, u_3) = 0. \tag{4.7}$$

In total,

$$\partial_{u_1} r(u_1, u_2) + \partial_{u_2} r(u_1, u_2) = [h(u_1) \otimes 1 + 1 \otimes h(u_2), r(u_1, u_2)]. \tag{4.8}$$

Define an invertible map $\psi(u)$ acting on $\mathfrak{g}$, such that

$$\frac{d}{du}\psi(u)[x] = \Big[h(u), \psi(u)[x]\Big] \qquad \forall\ x \in \mathfrak{g}, \tag{4.9}$$

and set

$$\widehat{r}(u_1, u_2) = [\psi^{-1}(u_1) \otimes \psi^{-1}(u_2)] r(u_1, u_2). \tag{4.10}$$

Since

$$\frac{d}{du}\psi^{-1}(u) = -\psi^{-1}(u)\left(\frac{d}{du}\psi(u)\right)\psi^{-1}(u), \tag{4.11}$$

(4.8) becomes

$$\partial_{u_1}\widehat{r}(u_1,u_2) + \partial_{u_2}\widehat{r}(u_1,u_2) = 0, \tag{4.12}$$

which shows that $\widehat{r}$ is of difference form. □

The importance of the two theorems above resides not only in their powerful classification of the possible classical integrable structures associated to simple Lie algebras, but also in how this structure turns out to determine quite uniquely the possible quantisations one can extract. This poses strong constraints on the possible types of infinite-dimensional quantum groups, thereby restricting the classes of quantum integrable systems one can ultimately realise.

Mathematically, the quantisation procedure involves the concept of *Lie bialgebras* and the so-called *Manin triples* (see for example [31] and references therein). The term *quantisation* incorporates the meaning of completing the classical algebraic structure to a quantum group, or, equivalently, obtaining from a classical r-matrix a solution to the *quantum Yang-Baxter Equation*

$$R_{12}\,R_{13}\,R_{23} \;=\; R_{23}\,R_{13}\,R_{12}, \qquad R_{ij} \sim 1\otimes 1 \,+ i\,\hbar\, r_{ij} + \mathcal{O}(\hbar^2). \tag{4.13}$$

The quantisation of the Sklyanin exchange relations is attained by simply *"completing the $\hbar$ series"* into the famous RTT relations, which will be discussed at length during this school:

$$\widehat{T}_1(u)\widehat{T}_2(u')R(u-u') \;=\; R(u-u')\widehat{T}_2(u')\widehat{T}_1(u), \qquad \widehat{T}(u) = T(u) + \mathcal{O}(\hbar) \tag{4.14}$$

where the quantum monodromy $\widehat{T}$ is now understood as the normal-ordering of the classical product integral expression. We can see that (4.14) tends to (3.35) for $\hbar \to 0$.

The associated quantum groups emerging from this quantisation process are then classified as *elliptic quantum groups* ($dim(\Gamma) = 2$), *quantum affine algebras* ($dim(\Gamma) = 1$), and *Yangians* ($\Gamma = \{0\}$, respectively)[15]. We refer to the lectures by F. Loebbert at this school [32] for more details on this.

This is then indeed a mathematical framework for transitioning from the classical to the quantum regime of the physics:

$$\{A,B\} = \lim_{\hbar\to 0}\frac{[A,B]}{i\,\hbar}. \tag{4.15}$$

One could say that for integrable systems one has an explicit exact formula for the r.h.s. of (4.15) *as a function of $\hbar$*. In a sense, the Sklyanin exchange relations are the best starting point wherefrom to quantise the theory, in a way that keeps the integrable structure manifest at every step.

[15] A theorem [30] says that, if $r^3 = 0$, then $R = e^r$ solves the Yang-Baxter equation. In general, R is a very complicated expression, which is the correspondent of the highly non-trivial procedure of quantising a classical Lie bialgebra [30, 31].

4.2. Analytic properties

In this section, we discuss the analytic properties of the classical r-matrix as a function of the complex spectral parameters.

Example

- A convenient way of displaying the connection between the classical r-matrix and the associated quantum group is the case of Yangians. Let us consider the so-called *Yang's r-matrix*:

$$r = \kappa \frac{\mathcal{C}_\otimes}{u_2 - u_1}. \tag{4.16}$$

This turns out to be the prototypical rational solution of the CYBE. Indeed, by definition of the Casimir $\mathcal{C}_\otimes$, one has $[\mathcal{C}_\otimes, t^a \otimes \mathbb{1} + \mathbb{1} \otimes t^a] = 0 \; \forall \, a$, and one can easily prove that (4.16) solves the CYBE.

This classical r-matrix is the one relevant for the non-linear Schrödinger model of example (3.21), as it was proven by Sklyanin [9]. Using this fundamental result, it is then an easy exercise to show that, combining (4.16) and (3.35) with (3.25), one obtains in particular that $\{a(u), a(u')\} = 0$, which was used in footnote 8.

As a matter of fact, upon quantisation the NLS model is found to conserve the particle number, and in each sector of the Fock space with fixed number of particles it reduces to a quantum mechanical problem with mutual delta-function interactions [9, 33]. This was exactly the context where Yang was working [34] when he came across the solution (4.16). In [9], Sklyanin went on to demonstrate that normal-ordering effects quantise the classical r-matrix (4.16) into the canonical Yangian R-matrix (in suitable units):

$$R = \mathbb{1} \otimes \mathbb{1} + \frac{i\kappa}{u_2 - u_1} \, \mathcal{C}_\otimes,$$

solution of the quantum Yang-Baxter equation.

One can expand the classical r-matrix (4.16) as follows:

$$\frac{r}{\kappa} = \frac{\mathcal{C}_\otimes}{u_2 - u_1} = \frac{\sum_a t_a \otimes t_a}{u_2 - u_1} = \sum_a \sum_{n \geqslant 0} t_a u_1^n \otimes t_a u_2^{-n-1} = \sum_a \sum_{n \geqslant 0} t_{a,n} \otimes t_{a,-n-1},$$

where we have assumed $|\frac{u_1}{u_2}| < 1$ for definiteness. Now we are capable of attributing the dependence on the spectral parameter u_1 (respectively, u_2) to generators in the first (respectively, second) space. This allows us to interpret the formula (4.16) as the representation of an r-matrix, which is an abstract object living in the tensor product $\mathcal{A}_{u_1}[\mathfrak{g}] \otimes \mathcal{A}_{u_2}[\mathfrak{g}]$ of two copies of a larger algebra $\mathcal{A}_u[\mathfrak{g}]$ constructed out of $\mathfrak{g}$. The assignement

$$t_{a,n} \; = \; u^n \, t_a \tag{4.17}$$

in (4.17) produces

$$[t_{a,m}, t_{b,n}] = \sum_c f_{abc} \, t_{c,m+n} \tag{4.18}$$

in terms of the structure constants f_{abc}. The relations (4.18) then identify in this case the algebra $\mathcal{A}_u[\mathfrak{g}]$ as the *loop algebra*[16] $\mathcal{L}_u[\mathfrak{g}]$ associated to $\mathfrak{g}$.

One can then make sense of the operation of *abstracting* the relations (4.18) away from the specific representation (4.17) they are originally seen to emerge from. Using solely these commutation relations, one can then verify that the abstract formal expression[17]

$$r = \sum_a \sum_{n \geqslant 0} t_{a,n} \otimes t_{a,-n-1} \tag{4.21}$$

provides a consistent classical r-matrix independently of specific representations of (4.18). In turn, the universal enveloping algebra $U\big(\mathcal{L}_u[\mathfrak{g}]\big)$ of the loop algebra $\mathcal{L}_u[\mathfrak{g}]$ is nothing else but the classical limit of the Yangian $\mathcal{Y}(\mathfrak{g})$:

$$\mathcal{Y}(\mathfrak{g}) \to U\big(\mathcal{L}_u[\mathfrak{g}]\big) \qquad \text{as} \quad \hbar \to 0. \tag{4.22}$$

Theorem. *The spans of the generators appearing separately on each factor of r must form two Lie subalgebras of $\mathfrak{g}$.*

Proof. By writing $r = \sum_{ab} r_{ab}(u)\, z^a \otimes z^b$, with the z's being a subset of the t's, one has that near the pole at $u_1 = u_2$ the CYBE reduces to

$$\sum_{abcd} \frac{c_{ab}(u_1)}{u_1 - u_2}\, r_{cd}(u_1 - u_3)\, ([z_a, z_c] \otimes z_b \otimes z_d + z_a \otimes [z_b, z_c] \otimes z_d) = 0,$$

with some function $c_{ab}(u_1)$. This implies

$$[z_a, z_c] = \sum_d f_{acd} z_d$$

for a subset of the structure constants f_{acd}. In [27], the Jacobi identity is shown, which proves that the two spans discussed above form Lie subalgebras of $\mathfrak{g}$. □

[16] A generalisation of the loop algebra is the so-called *affine Kac-Moody algebra* $\hat{\mathfrak{g}}$, associated to a finite-dimensional Lie algebra $\mathfrak{g}$. To obtain such a generalisation, one allows for a non-trivial central extension c. If we denote the generators of the affine Kac-Moody algebra as $s_{a,n} \equiv s_a \otimes v^n$ in terms of a formal parameter v, we can then write the defining relations of $\hat{\mathfrak{g}}$ as

$$[s_a \otimes v^n, s_b \otimes v^m] = [s_a, s_b] \otimes v^{n+m} + (s_a, s_b)\, n\, \delta_{n,-m}\, c, \tag{4.19}$$

with $(,)$ the scalar product induced on $\mathfrak{g}$ by the Killing form. One usually adjoins a derivation d to the algebra:

$$[d, s_a \otimes v^n] = n\, s_a \otimes v^n, \qquad d \equiv v \frac{d}{dv}, \tag{4.20}$$

in order to remove a root-degeneracy (see *e.g.* [35]).

[17] There are subtleties one needs to be careful about, regarding the convergence of formal series such as (4.21). An appropriate setting where to discuss such issues is typically provided by the so-called *p-adic topology* and *Poincaré-Birkhoff-Witt* bases.

These two span subalgebras together with $\mathfrak{g}$ form the so-called *Manin triple*. Characterisation of such a triple is an essential pre-requisite to identify the actual algebra, the quantum group is going to be built upon. In fact, suppose we had started with an r-matrix, solution of the classical Yang-Baxter equation, such that, however, none of the three requirements $(i) - (iii)$ in the Belavin-Drinfeld I theorem held. In particular, this could be because we have identified that $r \in \mathfrak{g} \otimes \mathfrak{g}$, but some of the basis element t_a never actually appear in r, causing the determinant $\det r_{ab}$ to vanish by a row of zeroes. The theorem we have just proven reassures us that we can always find a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $r \in \mathfrak{g}' \otimes \mathfrak{g}'$. For this restriction, r has now a chance of being non-degenerate.

Let us finally mention that classical r-matrices are core objects in the theory of quantum groups and deformation quantisation, and play a special role in the study of the so-called *Drinfeld double* (cf. F. Loebbert's lectures at this school [32]).

5. Solitons

In this section, we discuss the soliton solutions of integrable classical field theories. By way of introduction, let us report what is probably the archetype of solitons, contained in John Scott Russell's famous report of an event occurred at the Union Canal, Scotland, in 1834 [36]:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed".

In the language of integrable systems, we can model this situation using the famous Korteweg - de Vries (KdV) equation [37, 38] for a wave profile ϕ in shallow water[18]:

$$\partial_t \phi + \partial_x^3 \phi - 6 \phi \partial_x \phi = 0. \tag{5.1}$$

The KdV equation (5.1) admits, as a particular solution[19], a travelling soliton parameterised by two arbitrary real constants x_0 and v:

$$\phi = -\frac{v}{2} \operatorname{sech}^2 \left[\frac{\sqrt{v}}{2} (x - v\, t + x_0) \right]. \tag{5.2}$$

The Lax pair for the KdV equation is given by

$$L = \begin{pmatrix} 0 & -1 \\ u - \phi & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} -\phi_x & -4u - 2\phi \\ 4u^2 - 2u\,\phi + \phi_{xx} - 2\phi^2 & \phi_x \end{pmatrix}. \tag{5.3}$$

[18] This form of the equation gets mapped onto the one presented in S. Negro's lectures at this school, upon the identifications $\phi = -U, t = -t_3, x = w$, and compactification of x to $[0, 2\pi]$.

[19] A more general solution is obtained in terms of the Jacobi elliptic cosine function 'cn', hence the name *cnoidal wave*.

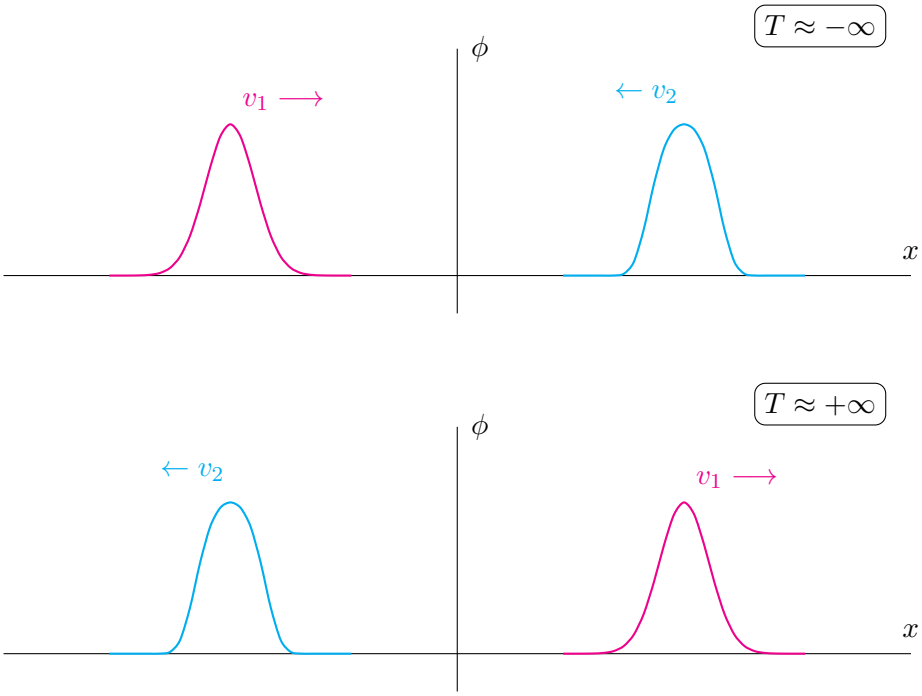

Figure 2: Scattering of solitons keeping their shapes and velocities.

The extra conservation laws of the integrable hierarchy prevent the waves from loosing their profiles throughout the time evolution. We will now discuss a very general method for solving integrable equations and find soliton solutions in a wide variety of cases.

5.1. The classical inverse scattering method

Let us show how Gardner, Green, Kruskal and Miura solved the KdV equation, with a method which since became a standard procedure for integrable partial differential equations [6]. As we pointed out in the Introduction, this method, dubbed of the *(classical) inverse scattering*, was afterwards adapted to quantum theories by the Leningrad school, and still to this day represents a paradigmatic approach to the quantisation of integrable systems. In the following section, we will show a more complicated application of the inverse scattering method to the Sine-Gordon equation.

The main feature that emerged from early numerical calculations performed on the KdV equation was that there are solutions describing multiple propagating profiles, which nevertheless scatter off each other preserving their individual shapes through the process (see Figure 2). This is quite surprising for a non-linear equation as the one at hand, and it is due to a perfect competition between the non-linearity $\phi\,\partial_x\phi$ (trying to concentrate the profile) and the dispersion $\partial_x^3\phi$ (trying to spread the profile). It also shows, in a way, how integrability is capable of restoring some features, which might be thought as rather pertaining to a linear behaviour, into a highly non-linear system.

Gardner *et al.* in *op. cit.* consider the auxiliary Schrödinger problem

$$\partial_x^2\,\psi = \Big(\phi - u(t)\Big)\psi, \tag{5.4}$$

with ϕ satisfying (5.1). Eq. (5.4) is equivalent to the first equation of our auxiliary linear problem (3.15), namely $\partial_x \Psi = L\Psi$, after one takes a further ∂_x on the latter, and then projects it onto the first vector component. Solving for ϕ in (5.4) for ψ not identically zero, and substituting back into (5.1) gives

$$(\partial_t u)\,\psi^2 + [\psi\, Q_x - \psi_x Q]_x = 0, \qquad Q = \partial_t\,\psi + \partial_x^3\,\psi - 3(\phi+u)\,\partial_x\psi. \tag{5.5}$$

One can see that, if ψ vanishes sufficiently fast at $|x| \to \infty$, integrating the first equation in (5.5) on the whole real line implies $\partial_t u = 0$, hence u is a constant *spectral parameter*. This means that we are effectively solving for the normalisable part of the spectral problem $\partial_x^2\,\psi = (\phi - u)\psi$. It also means that we are left with solving

$$[\psi\, Q_x - \psi_x Q]_x = 0, \qquad \text{i.e.} \qquad \psi\, Q_{xx} = Q\,\psi_{xx}, \tag{5.6}$$

from (5.5). It is then straightforward to check that, differentiating the equation

$$Q(x,t) = C(t)\,\psi + D(t)\,\psi \int^x \frac{dx}{\psi^2} \tag{5.7}$$

twice w.r.t. x, for two arbitrary integration constants $C(t)$ and $D(t)$, and re-using (5.7) once, one obtains (5.6).

At this point, we assume that ϕ vanishes at spatial infinity for any given time.

- The normalisable ψ modes, for $C = D = 0$, satisfy

  $$0 = Q(x,t) \to \partial_t\,\psi + \partial_x^3\,\psi - 3u\,\partial_x\psi \qquad \text{at } |x| \to \infty \tag{5.8}$$

  which is solved by the bound states

  $$\psi_n \to c_n\, e^{\pm 4k_n^3 t \mp k_n x} \qquad \text{at } x \to \pm\infty, \qquad k_n = \sqrt{-u_n},\ \ u_n < 0, \tag{5.9}$$

  expected to form the discrete part of the auxiliary spectral problem.

- We can extend our problem to the non-normalisable modes with a wave-like behaviour at spatial infinity, choosing u to be a constant. For this, we can first go back to (5.4) and deduce for instance, for $k^2 = u > 0$,

  $$\begin{aligned} &\psi \longrightarrow e^{-ikx} + b\, e^{ikx}, \qquad x \to \infty, \\ &\psi \longrightarrow a\, e^{-ikx}, \qquad x \to -\infty. \end{aligned} \tag{5.10}$$

  Solutions which are asymptotically plane waves at $x \to \pm\infty$ are called *Jost solutions*. Plugging this into (5.7), one finds as a solution $D = 0$, $C = 4ik^3$, and *scattering data* a, b determined as

  $$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)\, e^{8ik^3 t}. \tag{5.11}$$

  This is called the *direct scattering problem*.

Combining together all this information turns out to be sufficient to reconstruct ϕ. This means that we can reconstruct the potential ϕ in the auxiliary Schrödinger problem (5.4) from the scattering data. This is the *inverse scattering problem*. In fact, if $K(x,y)$, for $y \geqslant x$, is a solution of the *Gel'fand-Levitan-Marchenko* equation [39, 40]

$$K(x,y) + B(x+y) + \int_{-\infty}^{x} dz\, K(x,z)\, B(y+z) = 0, \tag{5.12}$$

where

$$B(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, b(k)\, e^{ikx} + \sum_n c_n^2 e^{-8k_n^3 t}\, e^{k_n x} \tag{5.13}$$

in terms of the coefficients b in (5.11) and c_n, k_n in (5.9), then one has

$$\phi = 2\frac{d}{dx} K(x,x). \tag{5.14}$$

The theory behind the Gel'fand-Levitan-Marchenko equation is deeply rooted in the technology which allows one to reconstruct the potential of a given Schrödinger problem from the knowledge of its reflection and transmission coefficients, which both feature in (5.13). The classical inverse scattering method is also regarded as a generalisation of the Fourier transform to non-linear problems. Let us sketch here an argument[20] that motivates formulae (5.12)-(5.14) in a simplified case. Consider the spectral problem

$$-\frac{\partial^2}{\partial t^2} w + \frac{\partial^2}{\partial x^2} w = V(x) w, \tag{5.15}$$

where $V(x)$ has a compact support $[-R, R]$ in the spatial direction x. This means that, in the regions $x < -R$ and $x > R$, w satisfies the free wave equation, hence we can write

$$\begin{aligned} w &= f_-(x-t) + g_-(x+t), \qquad x < -R, \\ w &= f_+(x-t) + g_+(x+t), \qquad x > R. \end{aligned} \tag{5.16}$$

Consider now two different solutions, characterised by the following asymptotics:

- *Case 1*

$$\begin{aligned} w &= \delta(x-t), \qquad t << -R, \\ w &= g_-(x+t) + f_+(x-t), \qquad t >> R. \end{aligned} \tag{5.17}$$

  For finite speed of propagation, this solution vanishes for $x > t$.

- *Case* 2

$$\begin{aligned} w &= \delta(x-t) + g_+(x+t), \qquad t << -R, \\ w &= f_+(x-t), \qquad t >> R. \end{aligned} \tag{5.18}$$

[20] A rigorous derivation of the Gel'fand-Levitan-Marchenko equation is beyond the scope of these lectures. For a wider context, the interested reader can for instance consult [41, 42]. Here, we found it convenient to follow a discussion by Terry Tao [43].

In both cases (which, in a sense, can be thought of as being dual to each other), the functions $f_\pm$ and $g_\pm$ appearing will be pictured as some sufficiently localised profiles. Let us give a special name to the function g_-, *i.e.*

$$g_-(z) \equiv B(z) \qquad \textit{"scattering data"}. \tag{5.19}$$

Let us focus on case 2, and make the ansatz

$$w = \delta(x-t) + K(x,t)\Theta(x-t) \tag{5.20}$$

for the full solution to (5.15), with Θ the Heaviside step function. Take K to vanish for $x < -R$. Plugging this ansatz back into (5.15) and collecting the terms proportional to $\delta(x-t)$, one gets

$$V(x) = 2\frac{d}{dx}K(x,x). \tag{5.21}$$

One then notices that, if $w(x,t)$ solves (5.15), so does $w(x,-t+s)$ for an arbitrary constant shift s. Therefore, also

$$w(x,t) + \int_{-\infty}^{\infty} ds\, B(s)\, w(x,-t+s) \tag{5.22}$$

does. If we use the ansatz (5.20), this means that

$$\delta(x-t) + K(x,t)\Theta(x-t) + B(x+t) + \int_{-\infty}^{x} ds\, B(s+t)\, K(x,s) \tag{5.23}$$

solves (5.15), and it coincides with $\delta(x-t) + B(x+t)$ when $x < -R$. Therefore, this solution corresponds to case 1 above, hence it must vanish for $x > t$. This in turn implies

$$K(x,t) + B(x+t) + \int_{-\infty}^{x} ds\, B(s+t)\, K(x,s) = 0. \tag{5.24}$$

The form we use in the case of the KdV equation basically involves the Fourier transform of the procedure we have just sketched.

As an example, the single-soliton solution of the KdV equation is obtained from the above procedure in the case when $b = 0$, and there is only one discrete eigenvalue u. In this situation, one simply has

$$B(x) = \gamma\, e^{kx}, \qquad \gamma \equiv c^2\, e^{-8k^3 t} \tag{5.25}$$

and it is therefore convenient to make an ansatz for K of the form

$$K(x,y) = K(x)\, e^{ky}. \tag{5.26}$$

Eq. (5.12) then becomes easy to solve after a simple integration for $k > 0$:

$$K(x) = -\frac{2\gamma\, k\, e^{kx}}{\gamma e^{2kx} + 2k}. \tag{5.27}$$

We immediately get from (5.14) that

$$\phi = -\frac{16\, c^2\, k^3\, e^{2k\left(-4k^2 t + x\right)}}{\left[c^2\, e^{2k\left(-4k^2 t + x\right)} + 2k\right]^2}. \tag{5.28}$$

For the choice $c^2 = 2k$, we obtain

$$\phi \;=\; -2\,k^2\,\mathrm{sech}^2\Big[k\big(-4k^2t + x\big)\Big], \tag{5.29}$$

which coincides with (5.2) for $v = 4k^2$. For further detail and a complete exposition, we recommend the lecture notes [44].

5.2. The Sine-Gordon equation and Jost solutions

We will now apply the inverse scattering method to the more complicated example of the Sine-Gordon theory (3.31). In this section, we will follow [1].

Let us look for solutions of (3.31) behaving at infinity like

$$\phi \to 0 \qquad \text{as } x \to -\infty, \qquad\qquad \phi \to \frac{q\pi}{\beta} \qquad \text{as } x \to +\infty. \tag{5.30}$$

The quantity q is called the *topological charge*. We will restrict to the case where q is an integer, which is compatible with the equation. It can be calculated as

$$q = \frac{\beta}{\pi}\int_{-\infty}^{\infty} dx\,\frac{\partial\phi}{\partial x} = \frac{\beta}{\pi}\Big[\phi(\infty) - \phi(-\infty)\Big]. \tag{5.31}$$

If $q = 1$ we speak of a soliton (or *kink*), while if $q = -1$ we speak of an anti-soliton (or *anti-kink*).

Direct Problem

We begin by focusing on what is called the *direct scattering problem*, namely, the first equation in (3.15):

$$\partial_x\Psi \;=\; L\,\Psi, \qquad \Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{5.32}$$

At infinity, the conditions (5.30) imply for (5.32)

$$\partial_x\Psi \;\to\; ik\,\sigma_1\Psi \qquad \text{as } x \to -\infty, \qquad\qquad \partial_x\Psi \;\to\; ike^{iq\pi}\,\sigma_1\Psi \qquad \text{as } x \to +\infty, \tag{5.33}$$

where σ_1 is a Pauli matrix, and

$$k = mu \;-\; \frac{m}{u}. \tag{5.34}$$

This means that the solution will behave as plane waves at infinity. Let us define

$$\bar\Psi = \begin{pmatrix}\bar\psi_1\\ \bar\psi_2\end{pmatrix} = \begin{pmatrix}-i\psi_2^*\\ i\psi_1^*\end{pmatrix}. \tag{5.35}$$

We will make the specific choice of two *Jost solutions*, determined by the following asymptotic behaviours;

$$\Psi_1 \;\to\; a\begin{pmatrix}1\\1\end{pmatrix}e^{ikx} - b\begin{pmatrix}1\\-1\end{pmatrix}e^{-ikx} \qquad x \to -\infty, \qquad\qquad \Psi_1 \;\to\; \begin{pmatrix}1\\e^{iq\pi}\end{pmatrix}e^{ikx} \qquad x \to +\infty,$$

$$\Psi_2 \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-ikx} \qquad x \to -\infty, \qquad \Psi_2 \to -b^* \begin{pmatrix} 1 \\ e^{iq\pi} \end{pmatrix} e^{ikx} + a \begin{pmatrix} e^{iq\pi} \\ -1 \end{pmatrix} e^{-ikx} \qquad x \to +\infty,$$

$$|a(u)|^2 \,+\, |b(u)|^2 \,=\, 1. \tag{5.36}$$

The last condition follows from considering that various Wronskians constructed with Ψ_i and $\bar{\Psi}_i$ are independent of x.

The two solutions Ψ_1 and Ψ_2 are independent as long as the Wronskian $\det|\Psi_1\,\Psi_2|$ does not vanish, namely, as long as the *Jost function* $a(u)$ does not have zeroes. Let us first put ourselves away from any zero of $a(u)$, and list a few properties of the two independent Jost solutions (without proof).

- The Jost solutions Ψ_1 and Ψ_2, when regarded as functions of u, are analytic in the upper half plane $\mathrm{Im}(u) > 0$.
- For a fixed x, the Jost solutions have the following asymptotics in u:

$$\Psi_1 \,\to\, e^{-i\frac{q\pi}{2}} \begin{pmatrix} e^{i\frac{\phi\beta}{2}} \\ e^{-i\frac{\phi\beta}{2}} \end{pmatrix} e^{ikx} \qquad |u| \to \infty, \qquad \Psi_1 \,\to\, e^{i\frac{q\pi}{2}} \begin{pmatrix} e^{-i\frac{\phi\beta}{2}} \\ e^{i\frac{\phi\beta}{2}} \end{pmatrix} e^{ikx} \qquad |u| \to 0,$$

$$\Psi_2 \,\to\, \begin{pmatrix} e^{i\frac{\phi\beta}{2}} \\ -e^{-i\frac{\phi\beta}{2}} \end{pmatrix} e^{-ikx} \qquad |u| \to \infty, \qquad \Psi_2 \,\to\, \begin{pmatrix} e^{-i\frac{\phi\beta}{2}} \\ -e^{i\frac{\phi\beta}{2}} \end{pmatrix} e^{-ikx} \qquad |u| \to 0.$$

- The Jost function $a(u)$ is analytic in the upper half plane $\mathrm{Im}(u) > 0$ and satisfies

$$\begin{aligned} &a(u) \,\to\, e^{-i\frac{q\pi}{2}} \qquad |u| \to \infty, \qquad a(u) \,\to\, e^{i\frac{q\pi}{2}} \qquad\qquad |u| \to 0, \\ &a(-u) \,=\, e^{-iq\pi}\, a^*(u) \qquad \text{for real } u. \end{aligned} \tag{5.37}$$

The two functions $a(u)$ and $b(u)$, the zeroes u_n of $a(u)$ and the proportionality constants c_n between Ψ_1 and Ψ_2 at the values u_n where the Jost solutions become dependent

$$\Psi_2(x, u_n) = c_n\, \Psi_1(x, u_n),$$

altogether form the set of *scattering data*.

The time evolution of the scattering data can now be explicitly computed by substitution into the *second* equation in (3.15). One gets

$$\begin{aligned} &a(u,t) = a(u,0), \qquad b(u,t) = e^{2im\left(u+\frac{1}{u}\right)t}\, b(u,0), \qquad \text{hence} \\ &u_n(t) = u_n(0), \qquad c_n(t) = e^{-2im\left(u_n+\frac{1}{u_n}\right)t}\, c_n(0). \end{aligned} \tag{5.38}$$

Inverse Problem

We now proceed to reconstruct the field ϕ from the solutions to the auxiliary linear problem. For this, we start by noticing that for real u the Jost solutions satisfy

$$\frac{1}{a(u)}\, e^{i\frac{q\pi}{2}\sigma_3}\, \widehat{\Psi}_2 \,=\, \xi(u)\, \widehat{\Psi}_1 \,+ i\, \bar{\widehat{\Psi}}_1, \tag{5.39}$$

where

$$\xi(u) = -\frac{b^*(u)}{a^*(u)}, \qquad \widehat{\Psi}_1 = e^{i\frac{q\pi-\beta\phi}{2}\sigma_3}\Psi_1, \qquad \widehat{\Psi}_2 = e^{-i\frac{\beta\phi}{2}\sigma_3}\Psi_1. \tag{5.40}$$

This can be obtained by using that, if $\psi(x,u)$ is a solution to the direct problem (5.32), so is $\sigma_3\,\psi(x,-u)$. One can then compare the asymptotic behaviours of $\bar{\Psi}_i(x,u)$ and $\sigma_3\Psi_i(x,-u)$, where Ψ_i are the two Jost solutions.

One can also prove that the (hatted) Jost solutions admit the following Fourier representation:

$$\begin{aligned}
\widehat{\Psi}_1 &= e^{ikx}\begin{pmatrix}1\\ e^{iq\pi}\end{pmatrix} + \int_x^\infty dy\left(v_1(x,y)+\frac{1}{u}w_1(x,y)\right)e^{iky},\\
\widehat{\Psi}_2 &= e^{-ikx}\begin{pmatrix}1\\ -1\end{pmatrix} + \int_{-\infty}^x dy\left(v_2(x,y)+\frac{1}{u}w_2(x,y)\right)e^{-iky},
\end{aligned} \tag{5.41}$$

for $v_i(x,y)$ and $w_i(x,y)$ two-component vectors with sufficiently regular behaviour.

The importance of the kernels v_i and w_i is that the knowledge of their explicit form allows to reconstruct the matrix L and therefore provides a solution for the original field ϕ. One in fact has that, plugging these expansions back into the direct problem,

$$e^{2i\beta\phi(x)} \;=\; \frac{im + e^{iq\pi}w_{1,2}(x,x)}{im + w_{1,1}(x,x)}, \tag{5.42}$$

where q is defined in (5.30) and $w_{i,j}$ is the component j of the vector w_i.

The final step of the process involves again the Gel'fand-Levitan-Marchenko equation for the kernels v_i and w_i appearing in the Fourier decomposition (5.41).

Theorem (Gel'fand-Levitan-Marchenko). *The kernels v_1, w_1 in (5.41) satisfy the following integral equations:*

$$\begin{aligned}
\bar{v}_1(x,y) \;&=\; f_0(x+y)\begin{pmatrix}1\\ e^{iq\pi}\end{pmatrix} + \int_x^\infty dz\left(f_0(z+y)\,v_1(x,z) + f_{-1}(z+y)w_1(x,z)\right),\\
\bar{w}_1(x,y) \;&=\; f_{-1}(x+y)\begin{pmatrix}1\\ e^{iq\pi}\end{pmatrix} + \int_x^\infty dz\left(f_{-1}(z+y)\,v_1(x,z) + f_{-2}(z+y)w_1(x,z)\right),
\end{aligned}$$

with the scattering data (5.38) entering these equations as

$$f_i(x) = -\frac{m}{2\pi i}\int_{-\infty}^\infty du\,u^i\,e^{ikx}\,\xi(u) + m\sum_n e^{ik(u_n)x}u_n^i m_n, \qquad m_n = \frac{c_n}{a'(u_n)}. \tag{5.43}$$

The Gel'fand-Levitan-Marchenko equations greatly simplify under the assumption that $b=0$, corresponding to absence of reflection in the auxiliary linear problem. The integral equation for v_1 reduces to a linear system. The simplest possible assumption is that there is only one purely imaginary zero $u_1 = i\mu_1$ of $a(u)$, with $\mu_1 < 0$, which collapses the entire system to one linear equation with straightforward solution. Plugging this solution into (5.42) returns the one-soliton solution

$$e^{i\beta\phi} = \frac{1+X}{1-X}, \qquad X = i\,e^{-2m\left(\mu_1+\frac{1}{\mu_1}\right)(x-x_0)+2m\left(\mu_1-\frac{1}{\mu_1}\right)t}, \qquad \mu_1 \equiv -\sqrt{\frac{1+v}{1-v}}, \tag{5.44}$$

which is a profile propagating with velocity $v = \frac{\mu_1^2-1}{\mu_1^2+1} \in [-1, 1]$. It is easy to see that the topological charge q in (5.30) equals 1 for this solution. Upon quantisation, solitons and anti-solitons become the elementary particles in quantum Sine-Gordon theory.

The other famous and slightly more complicated solution one obtains, is one that is normally quantised into a bound state of a soliton and an anti-soliton, and it accordingly has zero topological charge. This solution is called a *breather*. The profile for a (non-translating) breather is given by

$$\phi = \frac{2}{\beta} \arctan \frac{\sqrt{1-\omega^2}\, \cos(4m\omega t)}{\omega\, \cosh\left(4m\sqrt{1-\omega^2}x\right)}, \qquad \omega \in [0, 1]. \tag{5.45}$$

6. Conclusions

We hope that these lectures have stimulated the curiosity of the many young researchers present at this school to delve into the problematics of integrable systems, and have prepared the ground for the following lectures, where, in particular, the quantum version of integrability will be presented.

This field is constantly growing, attracting representatives of different communities with interests ranging from mathematics to mathematical physics and high-energy physics. We are sure that the new generation of physicists and mathematicians present here in Durham will accomplish great progress in all these avenues of investigation.

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No data beyond those presented and cited in this work are needed to validate this study.

References

[1] O. Babelon, D. Bernard and M. Talon, "Introduction to classical integrable systems," Cambridge University Press, 2003.

[2] L. D. Faddeev and L. A. Takhtajan, "Hamiltonian Methods in the Theory of Solitons," Berlin: Springer (1987)

[3] S. Novikov, S. V. Manakov, L. P. Pitaevsky and V. E. Zakharov, "Theory Of Solitons. The Inverse Scattering Method," New York, Usa: Consultants Bureau (1984) 276 P. (Contemporary Soviet Mathematics)

[4] G. Arutyunov, "Student Seminar: Classical and Quantum Integrable Systems", lectures delivered at Utrecht Univ., 2006-07 [http:// www.staff.science.uu.nl/ $\sim$ aruty101/ teaching.htm]

[5] C. Candu, M. de Leeuw, "Introduction to Integrability", lectures delivered at ETH, 2013, [http://www.itp.phys.ethz.ch/education/fs13/int]

[6] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, "Method for solving the Korteweg-deVries equation," Phys. Rev. Lett. **19** (1967) 1095.

[7] A. B. Shabat and V. E. Zakharov, "A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I," Funct. Anal. Appl. **8** (1974) 226.

[8] A. B. Shabat and V. E. Zakharov, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II," Funct. Anal. Appl. **13** (1979) 166.

[9] E. K. Sklyanin, "Quantum version of the method of inverse scattering problem," J. Sov. Math. **19** (1982) 1546 [*Zap. Nauchn. Semin.* **95** (1980) 55].

[10] L. D. Faddeev and V. E. Zakharov, "Korteweg-de Vries equation: A Completely integrable Hamiltonian system," Funct. Anal. Appl. **5** (1971) 280 [*Funkt. Anal. Pril.* **5N4** (1971) 18].

[11] V. G. Drinfeld, "Quantum groups," J. Sov. Math. **41** (1988) 898 [*Zap. Nauchn. Semin.* **155** (1986) 18].

[12] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized S-Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models," Annals Phys. **120** (1979) 253.

[13] R. J. Baxter, "Exactly solved models in statistical mechanics," Courier Corporation, 2007

[14] S. J. van Tongeren, "Introduction to the thermodynamic Bethe ansatz", in A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi, S. J. van Tongeren editors, "An Integrability Primer for the Gauge-gravity Correspondence", to appear in J. Phys. A.

[15] S. M. Ulam, "On combination of stochastic and deterministic processes - preliminary report," Bulletin of the American Mathematical Society, **53** (1947) 1120.

[16] R. S. Ward, "Integrable and solvable systems, and relations among them," Phil. Trans. Roy. Soc. Lond. A **315** (1985) 451.

[17] M. J. Ablowitz, S. Chakravarty and R. G. Halburd, "Integrable systems and reductions of the self-dual Yang-Mills equations," J. Math. Phys. **44** (2003) 3147.

[18] L. D. Faddeev, "How algebraic Bethe ansatz works for integrable model,", Les Houches lectures, 1995 [hep-th/9605187].

[19] A. Mironov, A. Morozov, Y. Zenkevich and A. Zotov, "Spectral Duality in Integrable Systems from AGT Conjecture," JETP Lett. **97** (2013) 45 [*Pisma Zh. Eksp. Teor. Fiz.* **97** (2013) 49] [arXiv:1204.0913 [hep-th]].

[20] M. Antonowicz and S. Rauch-Wojciechowski, "Lax representation for restricted flows of the KdV hierarchy and for the Kepler problem," Phys. Lett. A **171** (1992) 303.

[21] A. B. Shabat and V. E. Zakharov, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," Soviet Physics JETP **34** (1972) 62.

[22] L. A. Takhtajan, "Hamiltonian systems connected with the Dirac equation," Zapiski Nauchnykh Seminarov POMI **37** (1973) 66.

[23] J. M. Maillet, "Hamiltonian Structures for Integrable Classical Theories From Graded Kac-moody Algebras," Phys. Lett. B **167** (1986) 401.

[24] J. M. Maillet, "New Integrable Canonical Structures in Two-dimensional Models," Nucl. Phys. B **269** (1986) 54.

[25] N. J. MacKay, "On the classical origins of Yangian symmetry in integrable field theory," Phys. Lett. B **281** (1992) 90 [Phys. Lett. B **308** (1993) 444].

[26] F. Levkovich-Maslyuk, "Introduction to Bethe Ansatz", in A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi, S. J. van Tongeren editors, "An Integrability Primer for the Gauge-gravity Correspondence", to appear in J. Phys. A.

[27] A. A. Belavin and V. G. Drinfeld, "Solutions of the classical Yang-Baxter equation for simple Lie algebras", Funct. Anal. Appl. **16** (1982), 159

[28] A. A. Belavin and V. G. Drinfeld, "Triangle equation for simple Lie algebras", Mathematical Physics Reviews (ed. Novikov at al.) Harwood, New York (1984), 93

[29] A. A. Belavin and V. G. Drinfeld, "Classical Yang-Baxter equation for simple Lie algebras", Funct. Anal. Appl. **17** (1983), 220

[30] V. Chari and A. Pressley, "A guide to quantum groups," Cambridge, UK: Univ. Pr., 1994

[31] P. Etingof and O. Schiffman, "Lectures on Quantum Groups," Lectures in Mathematical Physics, International Press, Boston, 1998

[32] F. Loebbert, "Lectures on Yangian Symmetry", in A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi, S. J. van Tongeren editors, "An Integrability Primer for the Gauge-gravity Correspondence", to appear in J. Phys. A.

[33] H. B. Thacker, "Exact Integrability in Quantum Field Theory and Statistical Systems," Rev. Mod. Phys. **53** (1981) 253.

[34] C. N. Yang, "Some exact results for the many body problems in one dimension with repulsive delta function interaction," Phys. Rev. Lett. **19** (1967) 1312.

[35] P. Goddard and D. I. Olive, "Kac-Moody and Virasoro Algebras in Relation to Quantum Physics," Int. J. Mod. Phys. A **1** (1986) 303.

[36] J. S. Russell, "Report on Waves," Report of the fourteenth meeting of the British Association for the Advancement of Science, York, 1844 (London 1845), 311

[37] J. Boussinesq, *"Essai sur la theorie des eaux courantes, Memoires presentes par divers savants," Acad. des Sci. Inst. Nat. France*, XXIII, 1877

[38] D. J. Korteweg, G. de Vries, "On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves", Philosophical Magazine 39 (240) (1895), 422

[39] I. M. Gelfand and B. M. Levitan, "On the determination of a differential equation from its spectral function," American Mathematical Society, 1955.

[40] V. A. Marchenko, "Sturm-Liouville operators and their applications," *Kiev Izdatel Naukova Dumka* 1, 1977.

[41] F. Calogero, A. Degasperis, "Spectral Transform and Solitons I,", North-Holland, 1982

[42] S. Nettel, "Wave Physics. Oscillations - Solitons - Chaos,", Springer, 2003

[43] T. Tao, "Israel Gelfand", entry at https://terrytao.wordpress.com/2009/10/07/israel-gelfand/#more-2860

[44] M. Dunajski, "Integrable systems", University of Cambridge Lecture Notes, 2012 [http://www.damtp.cam.ac.uk/user/md327/teaching.html]

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Lectures on Yangian Symmetry

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Abstract

In these introductory lectures we discuss the topic of Yangian symmetry from various perspectives. Forming the classical counterpart of the Yangian and an extension of ordinary Noether symmetries, first the concept of nonlocal charges in classical, two-dimensional field theory is reviewed. We then define the Yangian algebra following Drinfel'd's original motivation to construct solutions to the quantum Yang–Baxter equation. Different realizations of the Yangian and its mathematical role as a Hopf algebra and quantum group are discussed. We demonstrate how the Yangian algebra is implemented in quantum, two-dimensional field theories and how its generators are renormalized. Implications of Yangian symmetry on the two-dimensional scattering matrix are investigated. We furthermore consider the important case of discrete Yangian symmetry realized on integrable spin chains. Finally we give a brief introduction to Yangian symmetry in planar, four-dimensional super Yang–Mills theory and indicate its impact on the dilatation operator and tree-level scattering amplitudes. These lectures are illustrated by several examples, in particular the two-dimensional chiral Gross–Neveu model, the Heisenberg spin chain and $\mathcal{N} = 4$ superconformal Yang–Mills theory in four dimensions. This review arose from lectures given at the Young Researchers Integrability School at Durham University (UK).

Contents

1 Introduction

> *"I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."* R. Feynman 1988 [1]

The possibility to grasp physical models, to efficiently compute observables and to explain mysterious simplifications in a given theory is largely owed to the realization of symmetries. In quantum field theories these range from discrete examples like parity, over spacetime Poincaré or super-symmetry to global and local internal symmetries. In the most extreme case, a theory has as many independent symmetries as it has degrees of freedom (possibly infinitely many). Roughly speaking, this is the defintion of an *integrable* model. The concept of integrability has many faces and can be realized or formulated in a variety of different and often equivalent ways. As we will see below, this symmetry appears in certain two- and higher-dimensional field theories or in quantum mechanical models like spin chains. While integrability in classical theories is rather well understood, quantum integrability still asks for a universal definition [2]. The nature of what we call a (quantum) integrable system can be identified by unveiling typical mathematical structures which have been subject to active research for many decades.

One realization of integrability is the *Yangian symmetry*, representing a generalization of Lie algebra symmetries in physics. This Hopf algebra was introduced by Vladimir Drinfel'd in order to construct solutions to the famous quantum Yang–Baxter equation [3–6]. Moreover, the Yangian algebra forms part of the familiy of quantum groups introduced by Drinfel'd and Michio Jimbo [3, 7, 8]. These provide the mathematical framework underlying the quantum inverse scattering method and the algebraic Bethe ansatz, which were developed by the Leningrad school around Ludwig Faddeev, see e.g. [9]. Hence, the Yangian represents a central concept within the framework of physical integrable models and their mathematical underpinnings.

The most common occurrence of Yangian symmetry in physics is the case of two-dimensional quantum field theories or discrete spin chain models. Here a global (internal) Lie algebra symmetry $\mathfrak{g}$ is typically enhanced to a Yangian algebra $Y[\mathfrak{g}]$. This Yangian combined with the Poincaré symmetry yields constraints on physical observables. These constraints following from the underlying Hopf algebra structure often allow to boostrap a quantity of interest, first of all the scattering matrix. One of the most prominent statements about symmetries of the S-marix is the famous four-dimensional Coleman–Mandula theorem [10]. It states that the spacetime and internal symmetries of the S-matrix may only be combined via the trivial direct product. Hence it is by no means obvious that an internal and a spacetime symmetry can be combined in a nontrivial way. In certain 1+1 dimensional field theories, however, it was shown that the Lorentz boost of the Poincaré algebra develops a nontrivial commutator with the internal Yangian generators. Thus, the internal and spacetime symmetry are coupled to each other [11, 12]. This interconnection implies stronger constraints on observables than a direct product symmetry, since the boost maps different representations of the Yangian to each other. That this nontrivial relation of the Yangian and the spacetime symmetry is possible can be attributed to the fact that the Yangian generators do not act on multi-particle states via a trivial tensor product generalization of their action on single particle states; they have a non-trivial coproduct, which violates the assumptions of the Coleman–Mandula theorem. Interestingly, the internal Yangian and the Poincaré algebra are linked in such a way that the Lorentz boost realizes Drinfel'd's automorphism of the Yangian algebra, which was originally designed to switch on the spectral parameter dependence of the quantum R-matrix.

The physical implementation of the abstract mathematical Yangian Hopf algebra can in fact be observed in the case of several interesting examples. A very intriguing physical system and a two-dimensional prime example in these lectures is the so-called chiral Gross–Neveu model [13]. This theory of interacting Dirac fermions ψ provides a toy model for quantum chromodynamics and features a plethora of realistic properties whose implementation by a simple Lagrangian is remarkable. In particular, the model has a conserved current of the form $j^\mu = \bar{\psi}\gamma^\mu\psi$. The local axial current given by $j^\mu_{\text{axial}} = \bar{\psi}\gamma^\mu\gamma_5\psi$ is not conserved in this model. Remarkably, however, it is possible to repair this property by adding nonlocal terms to the axial current, resulting in a conserved nonlocal current. Hence, one finds an additional hidden symmetry that is realized in a more subtle way than the naive local Noether current j^μ. Commuting the corresponding nonlocal conserved charges with each other, one finds an expression which is not proportional to either of the two original charges, but rather generates a new symmetry operator. Importantly, this procedure can be iterated, inducing more and more new generators and thereby an infinite symmetry algebra. As we will see, this algebra furnishes a realization of the Yangian and a way to formulate the integrability of this quantum field theory.

Another prominent occurence of Yangian symmetry is the case of integrable spin chain models. Here the action of the symmetry generators can be understood as a straightforward generalization of the above field theory operators to the case of a discrete underlying Hilbert space. Spin chains are typically defined by a Hamiltonian whose Yangian symmetry may be tested by commutation with the symmetry generators. Notably, the exact Yangian symmetry strongly depends on the particular boundary conditions of the system under consideration. While Yangian symmetry is exact on infinite spin chains (no boundaries), the symmetry is typically broken by periodic, cyclic or open boundary conditions.[1] Though this breaking implies that the spectrum is not organized into Yangian multiplets, the bulk Hamiltonian is still strongly constrained by requiring a vanishing commutator with the generators modulo boundary terms. Notably, the Lorentz boost of two-dimensional field theories can be generalized to the case of spin chain models, where the Poincaré algebra extends to the algebra containing all local conserved charges [14, 15]. These local charges furthermore allow to define generalized boost operators which in turn generate integrable spin chains with long-range interactions [16].

Interestingly, the above long-range spin chains play an important role in an a priory unexpected context, namely for a four-dimensional quantum field theory which represents another toy model for QCD. The planar maximally supersymmetric Yang–Mills theory in four dimensions[2] is a conformal gauge theory that is believed to be integrable. The Hamiltonian of this theory in form of the (asymptotic) dilatation operator maps to an integrable long-range spin chain Hamiltonian [17–19]. In consequence, the spectrum of local operators, i.e. the spectrum of this quantum field theory, can be obtained using the powerful toolbox of integrability in two dimensions. In fact, this Hamiltonian of a $\mathfrak{psu}(2,2|4)$ symmetric (the symmetry of the Lagrangian) spin chain features a bulk Yangian symmetry $Y[\mathfrak{psu}(2,2|4)]$ [20, 21].

Indications for the Yangian symmetry of $\mathcal{N} = 4$ superconformal Yang–Mills theory were found in the form of Ward identities for various 'observables'. In fact, also the *four-dimensional* S-matrix of the Yang–Mills theory features a Yangian symmetry. This can most clearly be seen on color-ordered tree-level scattering amplitudes [22] and extends to loop-level when including anomalous contributions into the symmetry equation [23–25]. Here the color order of scattering amplitudes plays an important role since it implements two-dimensional characteristics within

[1] The same applies to two-dimensional field theories which, however, are typically defined on the infinite line.

[2] This theory, further discussed in the main text, goes under the name planar $\mathcal{N} = 4$ superconformal Yang–Mills theory. Here $\mathcal{N} = 4$ refers to the number of supercharges. The planar limit corresponds to the limit $N \to \infty$ of an infinite number of colors of the SU(N) gauge symmetry.

this four-dimensional Yang–Mills theory. In consequence, the representation of the Yangian generators on the S-matrix resembles the representation on spin chains or the 2d S-matrix.

This review is published in a collection of lecture notes on integrability [26–30] introduced by [31]. The structure of the present lectures is as follows: In Section 2 we investigate how *classical* integrability makes an appearence in two-dimensional field theories, i.e. we discuss the classical analogue of Yangian symmetry. Then, in Section 3, we consider the Yangian algebra, its relation to the Yang–Baxter equation and its embedding into mathematical terminology. This section is more formal than the rest of the notes; in particular one may skip Section 3.2 and Section 3.3 without missing prerequisites for the subsequent sections. We continue by studying how Yangian symmetry is realized in two-dimensional *quantum* field theories, and we discuss some of the implications of the Yangian on the 2d scattering matrix. In Section 5 we consider the case of discrete spin chain models and point out similarities to the field theory case. Finally we introduce how Yangian symmetry plays a role in four-dimensional superconformal Yang–Mills theory. We finish with a summary and a brief outlook.

2 Classical Integrability and Non-local Charges in 2d Field Theory

In this section we briefly review how ordinary symmetries are related to conserved Noether currents in classical field theories. We will see that assuming the associated local current to be flat, we may construct additional nonlocal currents, which are also conserved. We investigate how these nonlocal currents relate to classical integrability and the Lax formalism. Finally, we consider the example of the Gross–Neveu model and comment on the implementation of nonlocal charges as Noether symmetries. The nonlocal charges considered in this section form the classical version of the Yangian [32, 33].

2.1 Local and Bilocal Symmetries

Consider a field theory with a Lagrangian $\mathcal{L}(\phi_A, \partial\phi_A)$. Here ϕ_A represents the fields of the theory, which we do not specify for the moment. Suppose the Lagrangian has a continous internal or spacetime symmetry which is infinitesimally realized by a variation $\delta\phi_A$, and for which the Lagrangian changes at most by a total derivative:

$$\delta\mathcal{L} = \partial_\mu f^\mu. \tag{2.1}$$

Via Noether's theorem this symmetry induces a *conserved current* j_μ which obeys the conservation law

$$\partial_\mu j^\mu = 0, \tag{2.2}$$

and takes the generic form

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\delta\phi_A(x) - f^\mu(\phi_A). \tag{2.3}$$

Depending on the symmetry, it can be convenient to expand the current in terms of the symmetry generators according to $j^\mu = j^\mu_a\, t_a$. Here the symmetry algebra $\mathfrak{g}$ is generated by the operators t_a which we assume to be anti-hermitian, i.e. $t_a = -t_a^\dagger$.[3] The generators obey the commutation relations

$$[t_a, t_b] = f_{abc}\, t_c, \tag{2.4}$$

[3]Here we think of an internal symmetry, e.g. SU(N).

and for simplicity of the displayed expressions, we refrain from distinguishing upper and lower adjoint indices $a, b, c, \ldots$.[4] The above conserved current gives rise to a *conserved charge* defined by the space integral over its time component[5]

$$\mathrm{J}(t) = \int \mathrm{d}^{d-1}x\, j^0(t,x). \tag{2.7}$$

Due to the conservation law (2.2) the conserved charge obeys the equation

$$\frac{\mathrm{dJ}(t)}{\mathrm{d}t} = -\int_V \mathrm{d}^{d-1}x\, \vec{\nabla}\cdot\vec{j}(t,x) = -\int_S d\vec{s}\cdot\vec{j}(t,x). \tag{2.8}$$

If we specify the considered situation to $d = 2$ spacetime dimensions, we find that the conserved charge obeys

$$\frac{\mathrm{dJ}(t)}{\mathrm{d}t} = j^1(t,S_-) - j^1(t,S_+), \tag{2.9}$$

where $S_\mp$ denotes the boundaries of space. We can now furthermore assume that the current falls off at the spatial boundaries, i.e.

$$j^\mu(t,x) \xrightarrow{x\to S_\pm} 0, \tag{2.10}$$

and thus the charge J is time independent: $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{J}(t) = 0$. In the following the canonical choice will be to consider an infinite volume V with $S_\pm \to \pm\infty$.

Lorentz boost. Consider a Lorentz transformation as an example of a Noether symmetry. Infinitesimally, this transformation can be represented by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu, \tag{2.11}$$

where $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$. For illustration, let us assume that we are dealing with scalar fields ϕ_A, on which the Lorentz transformation acts as

$$\phi_A(x) \to \phi_A(\Lambda^{-1}x) = \phi_A(x) - \lambda^\mu{}_\nu x^\nu \partial_\mu \phi_A(x). \tag{2.12}$$

Hence we have $\delta\phi_A = -\lambda^\mu{}_\nu x^\nu \partial_\mu \phi_A$. The Lagrangian then transforms according to

$$\delta\mathcal{L} = -\lambda^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\lambda^\mu{}_\nu x^\nu \mathcal{L}), \tag{2.13}$$

and the corresponding Noether current takes the form

$$j^\mu = -\lambda^\rho{}_\nu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)} x^\nu \partial_\rho\phi_A - \delta^\mu{}_\rho x^\nu \mathcal{L}\right] = -\lambda^\rho{}_\nu T^\mu{}_\rho x^\nu. \tag{2.14}$$

[4] In general, these indices are raised and lowered by the Killing form κ_{ab}, which, in a certain basis, is related to the structure constants and the algebra's dual Coxeter number $\mathfrak{c}_2$ via

$$\kappa_{ad} = f_{abc}f_{bcd} = \mathfrak{c}_2\,\delta_{ad}. \tag{2.5}$$

Alternatively, these algebraic quantities are often expressed in terms of the quadratic Casimir operator $\mathcal{C}$ in the adjoint representation:

$$-\mathcal{C}\delta_{ad} = f_{abc}f_{bcd} = (t_b^{\mathrm{adj}})_{ac}\,(t_b^{\mathrm{adj}})_{cd}, \tag{2.6}$$

and we have $\mathfrak{c}_2 = -\mathcal{C}$. In these notes we use either the symbol for the quadratic Casimir $\mathcal{C}$ or the dual coxeter number $\mathfrak{c}_2$ depending on the typical convention in the respective context.

[5] Often the (nonlocal) conserved charges are denoted by the letter $\mathcal{Q}$. Since the literature on integrability is full of $\mathcal{Q}$'s anyways, we will use the capital J here and save the $\mathcal{Q}$ for later.

Here $T^{\mu\nu}$ denotes the energy momentum tensor defined by

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \partial_\nu \phi_A - \delta^{\mu}{}_{\nu} \mathcal{L}. \tag{2.15}$$

Note that due to the arbitrariness of the infinitesimal transformation $\lambda^{\rho}{}_{\nu}$, the above current in d spacetime dimensions in fact contains $d(d-1)/2$ conserved quantities:

$$(j^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}, \tag{2.16}$$

which obey $\partial_\mu (j^\mu)^{\rho\sigma} = 0$. For $\rho, \sigma = i, j$ both being spatial indices, the Lorentz transformation corresponds to a rotation, while for $\rho, \sigma = 0, i$ being a combination of the time and one spatial component, the transformation represents a Lorentz boost. Since we are particularly interested in two spacetime dimensions, where only one single Lorentz transformation (a boost) exists, we consider the latter case which gives rise to a conserved charge of the form

$$\mathrm{J}^{0i} = \int \mathrm{d}^{d-1}x \, (x^0 T^{0i} - x^i T^{00}). \tag{2.17}$$

Note that if the fields have a non-trivial spin as opposed to the considered scalars, i.e. the fields transform non-trivially under the Lorentz group, an extra term has to be added to the above boost transformation. In the case at hand, we may take into account that the Hamiltonian density is defined as the 00-component of the energy-momentum tensor:

$$\mathrm{H}(x) = \pi^A(x)\dot{\phi}_A(x) - \mathcal{L}(x), \qquad\qquad \pi^A(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}_A}. \tag{2.18}$$

Moreover, since the above charge J^{0i} is conserved, its value is time-independent and we may simply choose $t = x^0 = 0$. Then, in $d = 1+1$ dimensions,[6] we can rewrite the above boost charge as the first moment of the Hamiltonian

$$\mathcal{B} \equiv \mathrm{J}_{01} = \int \mathrm{d}x \, x \, \mathrm{H}(x). \tag{2.19}$$

Suppose the above integral runs from S_- to S_+, such that we can formally write the conserved boost charge in the form of a bilocal integral given by[7]

$$\mathcal{B} \simeq \int_{S_-}^{S_+} \mathrm{d}x \int_{S_-}^{x} \mathrm{d}y \, \mathbb{1} \cdot \mathrm{H}(x) \equiv [\mathbb{1} \, | \mathrm{H}], \tag{2.20}$$

modulo a term $S_- \int_{S_-}^{S_+} \mathrm{d}x \, \mathrm{H}(x)$ which is proportional to the conserved energy and does hence not modify the property of the boost to be a conserved charge. Here $\mathbb{1} \equiv 1$ denotes the identity, cf. Figure 1.[8]

Note that the above example for a Noether charge deals with a spacetime symmetry. Below we will also encounter examples of internal symmetries and associated charges which may be extended to bilocal symmetries. The motivation for recalling the properties of the Lorentz boost here will become clear when we discuss the Yangian.

[6] We use the conventions $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1)$ and $\epsilon_{01} = 1$.

[7] For brevity we introduce the ordered product $[A|B] = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{x} \mathrm{d}y \, A(y)B(x)$.

[8] Note that the discarded term $S_- \int_{S_-}^{S_+} \mathrm{d}x \, \mathrm{H}(x)$ diverges in the limit $S_\pm \to \pm\infty$. For better readability we refrain here from antisymmetrizing the bilocal integral in order to regularize the expression. In Section 5.5 we will see that this formal bilocal expression $[\mathbb{1} \, | \mathrm{H}]$ composed of two local densities $\mathbb{1}$ and H takes a natural place in the class of bilocal charges with nontrivial densities on both of the bilocal legs.

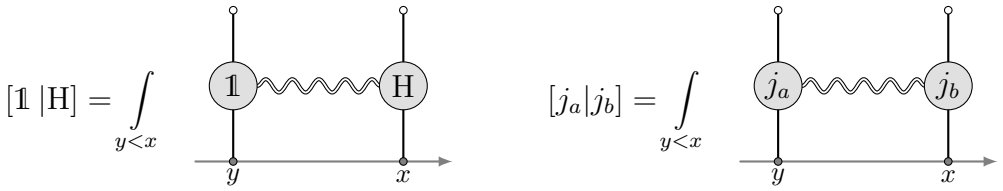


Figure 1: Left hand side: Boost generator written as bilocal integral. Right hand side: Bilocal operator composed of two charge densities.

Bilocal Symmetry. After having refreshed our memory about local symmetries, let us continue the survey on conserved currents and charges in 1+1 dimensions. Suppose the local current j^μ is not only conserved but also *flat*. Here flatness means that the current obeys the equation

$$[\partial_\mu + j_\mu, \partial_\nu + j_\nu] = 0, \tag{2.21}$$

i.e. it defines a flat connection. More explicitly, this can be written as

$$\partial_0 j_1 - \partial_1 j_0 + [j_0, j_1] = 0, \tag{2.22}$$

which for $j_\mu = j_{\mu a}\, t_a$ and $[t_a, t_b] = f_{abc}\, t_c$ reads in components

$$\partial_0 j_{1a} - \partial_1 j_{0a} + f_{abc}\, j_{0b} j_{1c} = 0. \tag{2.23}$$

Under the above flatness or zero-curvature condition, we may define an additional *bilocal* conserved current of the form

$$\widehat{j}^\mu_a(t,x) = \epsilon^{\mu\nu} j_{\nu a}(t,x) - \tfrac{1}{2} f_{abc}\, j^\mu_b(t,x) \int_{-\infty}^{x} \mathrm{d}y\, j^0_c(t,y), \tag{2.24}$$

which can be seen to be conserved modulo the conservation of the local current j^μ and the flatness condition:

$$\begin{aligned} \partial_\mu \widehat{j}^\mu_a(t,x) &= \partial_\mu \epsilon^{\mu\nu} j_{a\nu}(t,x) - \tfrac{1}{2} f_{abc} (\partial_\mu j^\mu_b(t,x)) \int_{-\infty}^{x} \mathrm{d}y\, j^0_c(t,y) + \tfrac{1}{2} f_{abc}\, \epsilon_{\mu\nu}\, j^\mu_b(t,x) j^\nu_c(t,x) \\ &= -\partial_0 j_{1a}(t,x) + \partial_1 j_{0a}(t,x) - [j_0(t,x), j_1(t,x)]_a = 0. \end{aligned} \tag{2.25}$$

We will refer to j^μ as the *level-zero* current and to $\widehat{j}^\mu$ as the *level-one* current. As for the local level-zero current, we can define a corresponding *level-one charge* by integration over the time component of the current:

$$\widehat{\mathrm{J}}_a(t) = \int_{-\infty}^{\infty} \mathrm{d}x\, \widehat{j}^0_a(t,x) = \int_{-\infty}^{\infty} \mathrm{d}x\, j^1_a(t,x) - \tfrac{1}{2} f_{abc} \int_{-\infty}^{\infty} \int_{-\infty}^{x} \mathrm{d}x\, \mathrm{d}y\, j^0_b(t,x) j^0_c(t,y). \tag{2.26}$$

The ordered one-dimensional integral has a similar form as (2.20), just that here both legs of the bilocal operator are nontrivial. Again we may write the charge in the compact form (cf. Figure 1)

$$\widehat{\mathrm{J}}_a(t) = \int_{-\infty}^{\infty} dx\, j^1_a(t,x) - \tfrac{1}{2} f_{abc}\, [j^0_b(t) | j^0_c(t)]. \tag{2.27}$$

Let us check explicitly under which conditions this charge is time independent. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{J}}_a(t) = -\int_{-\infty}^{\infty} \mathrm{d}x\, \partial_1 j_a^0(t,x) - f_{abc}\int_{-\infty}^{\infty} \mathrm{d}x\, j_b^0(t,x) j_c^1(t,x) \\ - \tfrac{1}{2} f_{abc}\int_{-\infty}^{\infty}\int_{-\infty}^{x} \mathrm{d}x\,\mathrm{d}y \Big[\big(\partial_1 j_{1b}(t,x)\big) j_c^0(t,y) + j_b^0(t,x)\big(\partial_1 j_{1c}(t,y)\big)\Big], \tag{2.28}$$

where we have used the flatness and conservation of the current. We can partially integrate to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{J}}_a(t) = j_a^0(t,-\infty) - j_a^0(t,\infty) - \tfrac{1}{2} f_{abc}\Big[j_b^0(t,\infty)\mathrm{J}_c - \mathrm{J}_b\, j_c^0(t,-\infty)\Big]. \tag{2.29}$$

Hence, as above in the discussion of the local charge conservation, we assume that (2.10)

$$j_a^0(t,x) \xrightarrow{x\to\pm\infty} 0, \tag{2.30}$$

such that indeed

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathrm{J}}_a(t) = 0. \tag{2.31}$$

Since the charges are time independent, we will no longer display their t-dependence in what follows. For the sake of compactness, we may also sometimes drop the explicit time dependence in the argument of the currents.

Notably, the above definition of the bilocal current distinguishes two points $S_\pm = \pm\infty$ in the one-dimensional space and thus allows for an order of the integration variables x and y. That this is an important input for the definition of the nonlocal charges can be realized by thinking about a possible generalization to the case of a compact periodic space which has no notion of order. It is also not obvious how to generalize the above definition of the nonlocal current to more than one space dimension.

Finally we note that the bilocal charge (2.26) is often written in the alternative and more symmetric forms

$$\widehat{\mathrm{J}}_a = \int_{-\infty}^{\infty} \mathrm{d}x\, j_a^1(t,x) - \tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y\, \theta(x-y)[j^0(t,x), j^0(t,y)]_a, \tag{2.32}$$

$$= \int_{-\infty}^{\infty} dx\, j_a^1(t,x) - \tfrac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y\, \epsilon(x-y)[j^0(t,x), j^0(t,y)]_a, \tag{2.33}$$

where θ denotes the step function and ϵ represents the sign function.

2.2 Nonlocal Charges and Lax Formulation

In the above section we have seen that two properties of the local current j^μ, namely to be conserved and flat, lead to a conserved bilocal current and an associated charge. Is this the only nonlocal charge we can construct from the above conditions? Let us understand things in a more systematical fashion along the lines of [34].[9]

Given a flat and conserved current j_μ, we can define a covariant derivative $D_\mu = \partial_\mu + j_\mu$. Conservation and flatness become the statements

$$[\partial_\mu, D^\mu] = 0, \qquad\qquad [D_\mu, D_\nu] = 0. \tag{2.34}$$

[9] Cf. also [35,36].

Now one may try an inductive approach. Suppose we have constructed a conserved current $j_\mu^{(n)}(x)$ of level n. The conservation implies that a function (the associated potential) $\chi^{(n)}(x)$ exists, for which

$$j_\mu^{(n)} = \epsilon_{\mu\nu}\partial^\nu \chi^{(n)}, \qquad n \geq 0. \tag{2.35}$$

In consequence, an additional current can be defined by

$$j_\mu^{(n+1)} = D_\mu \chi^{(n)}, \qquad n \geq -1, \tag{2.36}$$

where we set $\chi^{(-1)} = 1$. This current is conserved since we may use (2.34) to find

$$\partial^\mu j_\mu^{(n+1)} = \partial^\mu D_\mu \chi^{(n)} = D_\mu \partial^\mu \chi^{(n)} = \epsilon^{\mu\nu} D_\mu D_\nu \chi^{(n-1)} = 0, \qquad n \geq 0. \tag{2.37}$$

Here we have also used that (2.35) and (2.36) imply $\partial^\mu \chi^{(n)} = \epsilon^{\mu\nu} j_\nu^{(n)} = \epsilon^{\mu\nu} D_\nu \chi^{(n-1)}$ and that $\epsilon^{\mu\nu}[D_\mu, D_\nu] = -2[D_0, D_1] = 0$.

The start of the induction is $\chi^{(-1)} = 1$ with $j_\mu^{(-1)} = 0$ and such that $j_\mu^{(0)} = j_\mu$, which is indeed conserved by assumption. Then we can write

$$j_\mu^{(0)} = \epsilon_{\mu\nu}\partial^\nu \chi^{(0)}, \qquad\qquad \chi^{(0)} = -\int_{-\infty}^{x} \mathrm{d}y\, j^0(y). \tag{2.38}$$

and thus[10]

$$\widehat{j}_\mu \equiv j_\mu^{(1)} = D_\mu \chi^{(0)} = \epsilon_{\mu\nu} j^\nu(x) - j_\mu(x) \int_{-\infty}^{x} \mathrm{d}y\, j^0(y). \tag{2.40}$$

Hence, having shown the existence of a conserved current $j_\mu^{(0)} = j_\mu$ that obeys (2.34), one can construct $j_\mu^{(1)} = \widehat{j}_\mu$ and an infinite number of conserved nonlocal currents and consequently an infinite number of conserved nonlocal charges

$$\mathrm{J}^{(n)} = \int_{-\infty}^{+\infty} dx\, j_0^{(n)}(x). \tag{2.41}$$

The spectral parameter. Now we have obtained a set of conserved charges. Obviously, any linear combination of these charges will also furnish a conserved charge. We might thus wonder whether one can construct a conserved generating function $\mathrm{T}(u)$ whose expansion in u yields the conserved charges constructed above:[11]

$$\mathrm{T}(u) \simeq \sum_{k=-1}^{\infty} u^{-k-1} \mathrm{J}^{(k)}. \tag{2.42}$$

For the below discussion it may be useful to be familiar with some of the standard notions of classical integrability. These are for instance introduced in the review [26] or in the textbook [37]. Let us try to stay within the geometric picture that is suggested by the appearence of the

[10]Note that the conserved charge corresponding to this current equals the previous version up to level-zero charges since we have

$$\int_{-\infty}^{\infty} \mathrm{d}x\, j^0(x) \int_{-\infty}^{x} \mathrm{d}y\, j^0(y) = \tfrac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{x} \mathrm{d}x\, \mathrm{d}y\, [j^0(x), j^0(y)] + \tfrac{1}{4}\{\mathrm{J}, \mathrm{J}\}. \tag{2.39}$$

[11]Here we follow the usual convention and consider the expansion in $1/u$ instead of u and we set $\mathrm{J}^{(-1)} = 1$.

covariant derivative. In fact we may define a new covariant derivative $\mathcal{D}_\mu(u) = \partial_\mu - \mathrm{L}_\mu(u)$, where[12]

$$\mathrm{L}_\mu(t,x,u) = \frac{1}{u^2-1}\Big[j_\mu(t,x) + u\,\epsilon_{\mu\nu}j^\nu(t,x)\Big], \tag{2.43}$$

defines the *Lax connection* depending on the *spectral parameter* u. We may then collect both conditions in (2.34) by requiring that the following equation holds for all u:

$$[\mathcal{D}_\mu(u), \mathcal{D}_\nu(u)] = 0. \tag{2.44}$$

This furnishes a very compact way of writing the conservation and flatness conditions for the current j_μ. Note that we can understand the components of $\mathrm{L}_\mu(u)$ as a one-parameter family of Lax pairs, cf. [26].

The above equation (2.44) can be understood as a compatibility condition for the following so-called *auxiliary linear problem*

$$\mathcal{D}_\mu(u)\,\Phi(t,x) = 0, \tag{2.45}$$

which represents a system of two differential equations for the function $\Phi(t,x)$. In fact, applying another covariant derivative $\mathcal{D}_\nu(u)$ to this equation shows that the solution Φ is only well-defined, if (2.44) holds. Equation (2.45) relates an infinitesimal translation generated by ∂_μ to the flat connection $\mathrm{L}_\mu(u)$.

Next we determine the transport matrix $\mathrm{T}(t,x_0,x;u)$, which transports the solution $\Phi(t,x_0,u)$ along the interval $[x_0,x]$:

$$\Phi(t,x,u) = \mathrm{T}(t,x_0,x;u)\Phi(t,x_0,u). \tag{2.46}$$

Note that this transport matrix may be defined by the equations (cf. e.g. [35, 38, 39]):

$$\mathcal{D}_1(u)\,\mathrm{T}(t,x_0,x;u) = 0, \qquad\qquad \mathrm{T}(t,x_0,x_0;u) = 1. \tag{2.47}$$

We may integrate (2.47) along the x-coordinate and obtain the explicit path-ordered solution:

$$\mathrm{T}(t,x_0,x;u) = \mathcal{P}\,\exp\left[\int_{x_0}^{x} \mathrm{d}x'\,\mathrm{L}_1(t,x',u)\right]. \tag{2.48}$$

Here $\mathcal{P}$ denotes path-ordering with greater x to the left. Based on this expression, we define the *monodromy matrix*[13] $\mathrm{T}(t;u)$ as the transport matrix along the whole x-axis:

$$\mathrm{T}(t;u) \equiv \mathrm{T}(t,-\infty,\infty;u). \tag{2.49}$$

In order to evaluate the expansion of $\mathrm{T}(t;u)$ in powers of $1/u$, we note that for $v = 1/u$ we have

$$\mathrm{L}_\mu(t,x,v)\Big|_{v=0} = 0, \quad \tfrac{\mathrm{d}}{\mathrm{d}v}\mathrm{L}_\mu(t,x,v)\Big|_{v=0} = \epsilon_{\mu\nu}j^\nu(t,x), \quad \tfrac{\mathrm{d}^2}{\mathrm{d}v^2}\mathrm{L}_\mu(t,x,v)\Big|_{v=0} = 2j_\mu(t,x), \tag{2.50}$$

and thus we obtain

$$\mathrm{T}(t;u) = 1 - \frac{1}{u}\int_{-\infty}^{\infty}\mathrm{d}x\,j_0(t,x) + \frac{1}{u^2}\left[\int_{-\infty}^{\infty}\mathrm{d}x\,j_1(t,x) + \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{x}\mathrm{d}y\,j_0(t,x)j_0(t,y)\right] + \mathcal{O}\big(\tfrac{1}{u^3}\big). \tag{2.51}$$

[12] Note that in the language of differential forms, this is a linear combination of j and $\star j$, where $\star$ denotes the Hodge star. In this language the form of the Lax connection L might appear more natural.

[13] In ancient greek we have: *μόνος* [*"monos"*]: single and *δρόμος* [*"dromos"*]: course, path, racetrack.

Hence, we find indeed the level-zero and level-one charges as the first coeffcients of the expansion (2.42), cf. also (2.39). Assuming that $j_\mu(x) \xrightarrow{x\to\pm\infty} 0$, one can also show that in general

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{T}(t;u) = \mathrm{L}_0(t,+\infty,u)\mathrm{T}(t;u) - \mathrm{T}(t;u)\mathrm{L}_0(t,-\infty,u) \to 0. \tag{2.52}$$

That is the monodromy $\mathrm{T}(u) \equiv \mathrm{T}(t;u)$ really furnishes a conserved generating function for infinitely many conserved charges $\mathrm{J}^{(n)}$. See [26] for more details on the Lax formalism and the classical monodromy.

For certain models, the above nonlocal charges can be understood as the classical analogues of the Yangian algebra introduced below [32, 33]. Whether the charges really form a classical Yangian or another algebra depends on the Poisson algebra of the currents which in turn depends on the model. A classical Yangian can for instance be found in the chiral Gross–Neveu model or the principal chiral model, cf. [32]. In these models it was also shown that the above boost charge (2.19) Poisson-commutes with the charges J_a and $\widehat{\mathrm{J}}_a$:

$$\{\mathcal{B}, \mathrm{J}_a\} = 0, \qquad\qquad \{\mathcal{B}, \widehat{\mathrm{J}}_a\} = 0. \tag{2.53}$$

In Section 4.2 we will see that these commutation relations become nontrivial in the quantum theory.

2.3 Chiral Gross–Neveu Model

Let us consider some of the above concepts for the case of the 1+1 dimensional chiral Gross–Neveu model. This theory introduced in 1974 by Gross and Neveu [13] represents the two-dimensional version of the four-dimensional Nambu–Jona–Lasinio model [40]. It furnishes a toy model for QCD with a surprisingly rich catalogue of features. While conformal at the classical level, masses are generated by quantum corrections. Furthermore the theory is asymptotically free and can be solved in the large-N limit, where N is the parameter of the global symmetry $\mathfrak{u}(N)$. Remarkably, the theory is also integrable which can be seen as follows.

Local and nonlocal currents. We consider the Lagrangian of the $\mathfrak{u}(N)$ symmetric *chiral* Gross–Neveu model[14]

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar\psi^\alpha (i\partial\!\!\!/)\psi_\alpha + \frac{g^2}{2}\bigg[\Big(\sum_{\alpha=1}^{N}\bar\psi^\alpha\psi_\alpha\Big)^2 - \Big(\sum_{\alpha=1}^{N}\bar\psi^\alpha\gamma_5\psi_\alpha\Big)^2\bigg], \tag{2.54}$$

with $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. The Dirac fermions are denoted by $\psi_{\alpha j}$ and $\bar\psi^\alpha_j = \psi_i^{\dagger\alpha}(\gamma^0)_{ij}$ with $i,j = 1,2$ and with fundamental or anti-fundamental $\mathfrak{u}(N)$ indices α, respectively. The two-dimensional gamma matrices in the Weyl representation take the form

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.55}$$

and obey the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The Lagrangian also has a chiral $\mathfrak{u}(1)$ symmetry

$$\psi_\alpha \to e^{i\theta\gamma_5}\psi_\alpha, \tag{2.56}$$

[14]Note that there is also the $\mathfrak{o}(2N)$ symmetric *Gross–Neveu model* (without *chiral*) on the market, whose Lagrangian is given by dropping the γ_5-term.

which is not broken at the quantum level since the massive particles generated by spontaneous symmetry breaking are not charged under this symmetry, and the particles carrying a chiral charge decouple.[15]

Alternatively, the above Lagrangian can be written in the form

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/)\psi + g^2\Big[\big(\bar{\psi}\gamma_\mu t_a\psi\big)\big(\bar{\psi}\gamma^\mu t_a\psi\big)\Big], \tag{2.57}$$

where we do not display the sum over double indices $a, b, \cdots = 1, \ldots, N^2$ and $\alpha, \beta, \cdots = 1, \ldots, N$ from now on. Here $t_a = -t_a^\dagger$ represent the N^2 generators of $\mathfrak{u}(N)$. In the following we will refer to (2.57) as the chiral Gross–Neveu Lagrangian. For practical reasons one sometimes considers the case of generators t_a of $\mathfrak{su}(N)$ instead of $\mathfrak{u}(N)$.

The equivalence of the above Lagrangians can be shown by using the Fierz identity

$$(\gamma_\mu)_{ij}(\gamma^\mu)_{kl} = \delta_{il}\delta_{kj} - (\gamma_5)_{il}(\gamma_5)_{kj}, \tag{2.58}$$

as well as the following identity for the $\mathfrak{u}(N)$ generators:[16]

$$(t_a)_\alpha{}^\beta (t_a)_\gamma{}^\delta = -\tfrac{1}{2}\delta_\gamma^\beta\delta_\alpha^\delta. \tag{2.59}$$

The (Euler–Lagrange) equations of motion read

$$0 = i\partial_\mu\bar{\psi}^\alpha\gamma^\mu - 2g^2(\bar{\psi}\gamma^\mu t_a\psi)(\bar{\psi}\gamma_\mu t_a)^\alpha, \qquad 0 = i\gamma^\mu\partial_\mu\psi_\alpha + 2g^2(\gamma_\mu t_a\psi)_\alpha(\bar{\psi}\gamma^\mu t_a\psi). \tag{2.60}$$

Now we multiply these equations by ψ and $\bar{\psi}$, respectively, and use again the identity (2.59). Combining the two equations of motion then yields

$$i(\partial_\mu\bar{\psi}^\alpha)\gamma^\mu\psi_\beta + i\bar{\psi}^\alpha\gamma^\mu\partial_\mu\psi_\beta = 0, \tag{2.61}$$

which directly implies that the following current is conserved [42]:

$$j_a^\mu = -2g^2 i(\bar{\psi}^\alpha\gamma^\mu(t_a)_\alpha{}^\beta\psi_\beta). \tag{2.62}$$

Here the normalization is chosen for later convenience. In order to see the *flatness* of this current, we note that the equations of motion imply

$$\epsilon^{\mu\nu} i\partial_\mu(\bar{\psi}^\alpha\gamma_\nu\psi_\beta) = 2g^2\epsilon^{\mu\nu}(\bar{\psi}^\alpha\gamma_\mu\psi_\gamma)(\bar{\psi}^\gamma\gamma_\nu\psi_\beta), \tag{2.63}$$

where we used that $\{\gamma_5, \gamma_\mu\} = 0$ and $\gamma_\mu\gamma_5 = -\epsilon_{\mu\nu}\gamma^\nu$ as well as the identity (2.59). In terms of the current and contracting with a generator t_a, this takes the form

$$\epsilon^{\mu\nu}\partial_\mu(j_\nu)^\alpha{}_\beta\,(t_a)_\alpha{}^\beta = \epsilon^{\mu\nu}(j_\mu)^\alpha{}_\gamma(j_\nu)^\gamma{}_\beta\,(t_a)_\alpha{}^\beta, \tag{2.64}$$

and thus yields the flatness condition

$$\partial_0 j_{1a} - \partial_1 j_{0a} + [j_0, j_1]_a = 0. \tag{2.65}$$

In consequence, we can construct a bilocal current $\widehat{j}$ according to the procedure described above.

[15] Therefore this mass generation mechanism is not in contradiction with Coleman's theorem forbidding Goldstone bosons in two dimensions [41].

[16] For $\mathfrak{su}(N)$ symmetry the Lagrangian (2.54) gets an extra $1/N$ term coming from the $\mathfrak{su}(N)$ identity $(t_a)_\alpha{}^\beta(t_a)_\gamma{}^\delta = -\frac{1}{2}\delta_\gamma^\beta\delta_\alpha^\delta + \frac{1}{2N}\delta_\alpha^\beta\delta_\delta^\gamma$. For a more transparent illustration of the equivalence of the two Lagrangians we have considered the $\mathfrak{u}(N)$ symmetric Lagrangian here.

Axial current. Note that as a starting point to obtain a bilocal current we might also have considered the axial current

$$(j_{\text{axial}})^\mu_a = -2g^2 i\bar{\psi}\gamma^5\gamma^\mu t_a\psi = \epsilon^{\mu\nu} j_{\nu a} \tag{2.66}$$

which is familiar from our quantum field theory course, but which is not conserved in this model since (cf. (2.65))

$$\partial_\mu (j_{\text{axial}})^\mu_a = \partial_\mu \epsilon^{\mu\nu} j_{\nu,a} = -\partial_0 j_{1a} + \partial_1 j_{0a} \neq 0. \tag{2.67}$$

However, the bilocal current constructed from the conserved current j^μ_a can be understood as a nonlocal completion of this axial current which is then conserved as seen above, cf. [43]:

$$\widehat{j}_{\mu a}(x) = (j_{\text{axial}})_{\mu a} - \tfrac{1}{2} \int_{-\infty}^{x} \mathrm{d}y\, [j_\mu(x), j_0(y)]_a. \tag{2.68}$$

Poisson algebra and Lax formalism. In order to study the symmetry algebra that is generated by the above currents, we have to define a Poisson bracket for the Dirac fermions [39]:

$$\{F, G\} = i \int \mathrm{d}x \sum_{\substack{\alpha=1 \\ j=1,2}}^{N} F\left(\frac{\overleftarrow{\delta}}{\delta\psi_j^{\dagger\alpha}(x)} \frac{\vec{\delta}}{\delta\psi_{\alpha,j}(x)} + \frac{\overleftarrow{\delta}}{\delta\psi_{\alpha,j}(x)} \frac{\vec{\delta}}{\delta\psi_j^{\dagger\alpha}(x)} \right) G. \tag{2.69}$$

Here the arrows are introduced to take care of the Graßmann statistics of the fields and they indicate whether the variation acts on the function F or G. Using this definition of the Poisson bracket one can show that the current (2.62) obeys the algebra relations

$$\{j^\mu_a(x), j^\nu_b(y)\} = 2g^2 \delta(x-y) f_{abc}\, j^{|\mu-\nu|}_c, \tag{2.70}$$

with the $\mathfrak{su}(N)$ structure constants f_{abc}. The Lax connection and monodromy matrix can be defined as in (2.43) and (2.48), respectively. Their commutators with the classical R-matrix of the chiral Gross–Neveu model (see e.g. [26, 37] for these notions of classical integrability)

$$r(u,v) = \frac{\mathcal{C}_\otimes}{u-v}\,, \tag{2.71}$$

may then be considered as the fundamental integrability equations of this physical system, cf. [39]. For $\mathfrak{g} = \mathfrak{u}(N)$ and generators t_a in the fundamental representation, the tensor Casimir is given by $\mathcal{C}_\otimes = \mathbb{P}$, with $\mathbb{P}$ representing the permutation operator that acts on a state $a \otimes b$ according to

$$\mathbb{P}\, a \otimes b = b \otimes a, \tag{2.72}$$

and on an operator $A \otimes B$ by conjugation:

$$\mathbb{P}\, A \otimes B\, \mathbb{P} = B \otimes A. \tag{2.73}$$

We will encounter the permutation operator in its role as the tensor Casimir several times in this review.

2.4 Nonlocal Symmetries as Noether Charges

A very valid question is whether also nonlocal symmetries can be understood as Noether symmetries. At least for particular cases this question has been answered with a 'yes', cf.

e.g. [44, 45]. For illustration let us briefly review some results of [44] and consider the so-called *principal chiral model* in two dimensions with Lagrangian

$$\mathcal{L} = \frac{1}{16} \operatorname{Tr} \partial_\mu g(x) \partial^\mu g^{-1}(x). \tag{2.74}$$

Here the field $g(x)$ is group-valued, i.e. an element of a group G. The equations of motion take the form of a conservation equation

$$\partial_\mu j^\mu = 0, \tag{2.75}$$

for the current

$$j_\mu \equiv g^{-1} \partial_\mu g = -(\partial_\mu g^{-1}) g. \tag{2.76}$$

This current is also flat. As discussed in [44], one may define the following nonlocal field variation

$$\delta^{(1)}_\rho g = -4g[\chi^{(0)}, \rho], \qquad \chi^{(0)}(x) = \tfrac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}y \, \epsilon(x-y) j_0(y), \tag{2.77}$$

where $\rho = t^a \rho^a$, with t^a denoting the generators of the group G and ρ^a being some constants. Here $\chi^{(0)}$ represents again the potential associated to the level-zero current of (2.38). The Lagrangian is invariant under this transformation up to a total derivative:

$$\delta^{(1)}_\rho \mathcal{L} = \tfrac{1}{2} \operatorname{Tr} j_\mu \partial^\mu [\chi^{(0)}, \rho] = \partial^\mu \tfrac{1}{2} \operatorname{Tr} \left[\left(\tfrac{1}{2} \epsilon_{\mu\nu} [\partial^\nu \chi^{(0)}, \chi^{(0)}] + \epsilon_{\mu\nu} j^\nu \right) \rho \right]. \tag{2.78}$$

Importantly, the equations of motion have not been used to arrive at this form. This level-one symmetry (cf. (2.3)) yields the conserved level-one Noether current

$$j^{(1)}_\mu = -\epsilon_{\mu\nu} j^\nu + [j_\mu, \chi^{(0)}] - \tfrac{1}{2} \epsilon_{\mu\nu} [\partial^\nu \chi^{(0)}, \chi^{(0)}]. \tag{2.79}$$

The conservation of this level-one current implies the flatness of the level-zero current, which is very much in agreement with our intuition gained in the previous subsections:

$$\partial^\mu j^{(1)}_\mu \simeq -\partial_0 j_1 + \partial_1 j_0 - [j_0, j_1]. \tag{2.80}$$

Interestingly, the current (2.79) does not have the standard form of (2.24). In fact, the current is conserved without making use of the equations of motion. It is thus conserved on the set of all fields, i.e. off-shell. Using the equations of motion such that

$$\partial_\mu \chi^{(1)} = -\epsilon_{\mu\nu} j^\nu, \tag{2.81}$$

(2.79) reduces to the standard form (2.24) of the level-one current. Note that one might also have started with an ansatz of the form (2.79) in order to determine $\chi^{(0)}$ such that $j^{(1)}$ is conserved, cf. [46]. Notably, the above symmetries may be extended to a one-parameter family of nonlocal Noether symmetries [45]. As the monodromy considered above, this family furnishes a generating function for the parameter independent symmetries. Before we discuss the physical realization of the quantum version of the classical nonlocal symmetries considered in the previous subsections, we will now introduce the Yangian.

3 The Yangian Algebra

This section follows the line of the beautiful original papers by Drinfel'd who introduced the notion of Yangians in the context of quantum groups. In 1990 he was awarded the Fields Medal for his work on quantum groups and for his work in number theory. We will discuss three different realizations of the Yangian, which means three different mathematical definitions of the same algebraic structure that are related by isomorphisms. As opposed to the rest of these notes, in this section we sometimes distinguish between abstract algebra elements, e.g. a generator J, and their representation, e.g. $\rho(\mathrm{J})$.

3.1 Yang's R-matrix and the First Realization

One of the most important concepts underlying integrable models in general is the famous quantum Yang–Baxter equation. This equation was found to emerge in the context of a one-dimensional scattering problem by Yang in 1967 as well as for the eight-vertex model by Baxter in 1972 [47, 48] (see also [49]). In fact also the Yangian was defined in order to determine solutions to this equation. Let us see how this happened.

Yang's solution to the Yang–Baxter equation. In the paper [47] (see also [50]) Yang considered the following one-dimensional Hamiltonian for n interacting particles in a delta-function potential:

$$H = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + 2c \sum_{1\le j<k\le n} \delta(x_j - x_k), \qquad c > 0. \tag{3.1}$$

He made a (coordinate) Bethe ansatz[17] (cf. [29]) for the wavefunction of this quantum mechanical problem, which in the domain $0 < x_{k_1} < \cdots < x_{k_n} < L$ takes the form

$$\Psi(x_{k_1} < \cdots < x_{k_n}) = \sum_{\{j_1,\ldots,j_n\}\in \mathrm{Perm}\{1,\ldots,n\}} M_{k_1,\ldots,k_n,j_1,\ldots,j_n} \exp i[p_{j_1}x_{k_1} + \cdots + p_{j_n}x_{k_n}], \tag{3.2}$$

with the sum running over all $n!$ permutations of $1,\ldots,n$. Here M can be organized as an $n!\times n!$ matrix spanned by the $n!$ column vectors ξ:

$$M = \left(\xi_{I_1}, \xi_{I_2}, \ldots, \xi_{I_{n!}}\right). \tag{3.3}$$

These vectors have indices $I_1 = \{1,2,3\ldots,n\}$, $I_2 = \{2,1,3,\ldots,n\}$, $\ldots$, $I_{n!} = \{n, n-1,\ldots,1\}$. Notably, with this general ansatz Yang made no assumption on the symmetries of the wavefunction or the exchange statistics of the particles, respectively. It is however assumed that the scattering is purely elastic, i.e. that the values of momenta form a fixed set and are conserved individually. Often, in addition a particular exchange symmetry is assumed which allows to reduce the matrix M in the above ansatz to one row.[18]

From the form of the Hamiltonian (3.1), one can deduce by integrating the Schrödinger equation in center of mass coordinates that the wavefunction Ψ has to be continuous at $x_j = x_k$, while its first derivative should have a discontinuity at these points. Yang found that these conditions are satisfied at for instance $x_{k_3} = x_{k_4}$ if the permutation of the momentum labels j_3 and j_4 is compensated by a factor of the so-called R-matrix:

$$\xi_{j_1,j_2,j_3,j_4,j_5\ldots j_n} = \mathbb{P}_{34}\,\mathrm{R}_{34}(u_{j_4j_3})\,\xi_{j_1,j_2,j_4,j_3,j_5,\ldots,j_n}. \tag{3.4}$$

Here we make the exchange operator $\mathbb{P}_{34}$ for the particles with coordinates x_{k_3} and x_{k_4} explicit, while it is sometimes included into an alternative definition of the R-operator.[19] The above R-matrix accounts for the scattering of two particles.

[17] The Bethe ansatz is named after Hans Bethe's solution to the Schrödinger equation for a spin chain [51].

[18] For identical fermions one would have $\Psi(x_{k_1},\ldots,x_{k_i},x_{k_j},\ldots,x_{k_n}) = -\Psi(x_{k_1},\ldots,x_{k_j},x_{k_i},\ldots,x_{k_n})$. For identical bosons the physical system with the Hamiltonian (3.1) is called the Lieb–Liniger model [52] and we would have $\Psi(x_{k_1},\ldots,x_{k_i},x_{k_j},\ldots,x_{k_n}) = \Psi(x_{k_1},\ldots,x_{k_j},x_{k_i},\ldots,x_{k_n})$.

[19] The operator $\mathbb{P}_{ij}$ represents the permutation operator on the vector ξ_I permuting the entries k_i and k_j. An alternative definition of the R-matrix found in the literature is $\check{\mathrm{R}}_{ij} = \mathbb{P}_{ij}\,\mathrm{R}_{ij}$ (note that $\mathbb{P}^2 = \mathbb{1}$). Acting on ξ, we have for identical bosons $\mathbb{P}_{ij} = 1$ while for identical fermions $\mathbb{P}_{ij} = -1$. For a model of identical bosons for instance, whose wavefunction is symmetric under exchange of particles at x_{k_3} and x_{k_4}, the permutation operator on the right hand side of (3.4) acts as the identity and $\mathrm{R}_{34}(u_{j_4j_3})$ represents the scattering matrix for the two bosonic particles 3 and 4 with momentum difference $u_{j_4j_3} = p_{j_4} - p_{j_3}$.

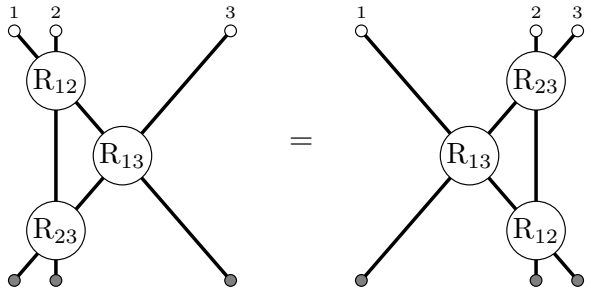


Figure 2: Illustration of the Yang–Baxter equation.

As discussed by Yang, the $n!(n-1)$ equations of the above form (3.4) are mutually consistent, if the R-matrix is unitary, i.e. if we have $\mathrm{R}_{\ell m}(u)\mathrm{R}_{m\ell}(-u) = 1$ and if the following *quantum Yang–Baxter equation* is obeyed, cf. Figure 1 (see e.g. [53] for a nice introduction to the Yang–Baxter equation by Jimbo):

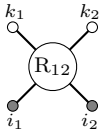
$$\mathrm{R}_{12}(u_{12})\mathrm{R}_{13}(u_{13})\mathrm{R}_{23}(u_{23}) = \mathrm{R}_{23}(u_{23})\mathrm{R}_{13}(u_{13})\mathrm{R}_{12}(u_{12}). \tag{3.5}$$

For three identical bosons for instance, $\mathbb{P}$ acts on ξ as the identity, and the Yang–Baxter equation can be understood by noting that via (3.4) the expression ξ_{321} can be obtained from ξ_{123} in two different ways, which have to be consistent:

$$\mathrm{R}_{12}(u_{12})\mathrm{R}_{13}(u_{13})\mathrm{R}_{23}(u_{23})\xi_{123} = \xi_{321} = \mathrm{R}_{23}(u_{23})\mathrm{R}_{13}(u_{13})\mathrm{R}_{12}(u_{12})\xi_{123}. \tag{3.6}$$

The quantum Yang–Baxter equation is of central importance for integrable models and appears in many different contexts. In general, it represents an operator equation acting on three spaces $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$ labeled $1, 2$ and 3. Each R-matrix (e.g. R_{12}) acts on two spaces (e.g. 1 and 2), and is a four-index object more explicitly written as[20]

$$\mathrm{R}^{k_2k_1}_{i_1i_2} = [\mathrm{R}_{12}]^{k_2k_1}_{i_1i_2} = \quad .$$



(3.8)

That is, when acting on n-dimensional vector spaces $\mathbb{V}$ with basis vectors $v_1, \dots, v_n$ we have[21]

$$\mathrm{R}(u)[\, v_i \otimes v_j] = \sum_{k,l} \mathrm{R}^{kl}_{ij}(u)\, v_k \otimes v_l. \tag{3.10}$$

Coming back to the above specific model with delta-function potential, the solution to the quantum Yang–Baxter equation given by Yang takes the form

$$\mathrm{R}_{\ell m}(u_{ij}) = \frac{u_{ij}}{u_{ij} + ic}\Big(\mathbb{1}_{\ell m} - \frac{ic}{u_{ij}}\, \mathbb{P}_{\ell m} \Big), \tag{3.11}$$

[20] Alternatively, one can write the Yang–Baxter equation as

$$\mathrm{R}^{k_2k_1}_{j_1j_3}(u_{12})\mathrm{R}^{k_3j_3}_{j_2i_3}(u_{13})\mathrm{R}^{j_2j_1}_{i_1i_2}(u_{23}) = \mathrm{R}^{k_3k_2}_{j_3j_2}(u_{23})\mathrm{R}^{j_3k_1}_{i_1j_1}(u_{13})\mathrm{R}^{j_2j_1}_{i_2i_3}(u_{12}). \tag{3.7}$$

[21] Using a tensor product notation, the R-matrices entering the Yang–Baxter equation can also be written as

$$\mathrm{R}_{12} = \mathrm{R} \otimes \mathbb{1}, \qquad \mathrm{R}_{23} = \mathbb{1} \otimes \mathrm{R}, \qquad \mathrm{R}_{13} = (\mathbb{P} \otimes \mathbb{1})\mathrm{R}_{23}(\mathbb{P} \otimes \mathbb{1}). \tag{3.9}$$

where the parameter $u_{ij} = p_i - p_j$ is given by the difference of the particle momenta. Note again that for instance for the symmetry algebra $\mathfrak{u}(N)$ with generators J_a in the fundamental representation, the permutation operator can be written as the tensor Casimir operator $\mathbb{P} = \mathcal{C}_\otimes = \mathrm{J}_a \otimes \mathrm{J}_a$. For this reason, the solution

$$\mathrm{R}(u) = \mathbb{1} + \frac{c}{u}\,\mathcal{C}_\otimes, \tag{3.12}$$

to the quantum Yang–Baxter equation is called *Yang's R-matrix*. Here c denotes some constant.

The Yangian. Almost twenty years after Yang, in 1985, Drinfel'd studied the quantum Yang–Baxter equation in order to develop an efficient method for the construction of its solutions [3]. Drinfel'd was one of the pioneers in introducing the related concept of quantum groups which he motivates as follows:

> *"Recall that both in classical and in quantum mechanics there are two basic concepts: state and observable. In classical mechanics states are points of a manifold M and observables are functions on M. In the quantum case states are 1-dimensional subspaces of a Hilbert space H and observables are operators in H (we forget the self-adjointness condition). The relation between classical and quantum mechanics is easier to understand in terms of observables. Both in classical and in quantum mechanics observables form an associative algebra which is commutative in the classical case and non-commutative in the quantum case. So quantization is something like replacing commutative algebras by noncommutative ones."* V. Drinfel'd 1986 [5].

For Drinfel'd the starting point to understand the quantum R-matrix $\mathrm{R}(u,\hbar)$ was its classical counterpart $r(u)$ obtained in the limit $\hbar \to 0$ from

$$\mathrm{R}(u,\hbar) \simeq \mathbb{1} + \hbar\, r(u) + \mathcal{O}(\hbar^2). \tag{3.13}$$

Subject to the quantum Yang–Baxter equation (3.5), the classical R-matrix $r(u)$ satisfies the classical Yang–Baxter equation:

$$[r_{12}(u_{12}), r_{13}(u_{13})] + [r_{12}(u_{12}), r_{23}(u_{23})] + [r_{13}(u_{13}), r_{23}(u_{23})] = 0. \tag{3.14}$$

Drinfel'd considered Yang's solution

$$r(u) = \frac{1}{u}\mathcal{C}_\otimes = \frac{1}{u}\mathrm{J}_a \otimes \mathrm{J}_a \tag{3.15}$$

of the classical Yang–Baxter equation. Here $\mathcal{C}_\otimes = \mathrm{J}_a \otimes \mathrm{J}_a$ again represents the tensor Casimir operator of the underlying finite dimensional simple Lie algebra $\mathfrak{g}$ with generators J_a. Given a representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ of the Lie algebra, Drinfel'd's intention was to show that solutions to the quantum Yang–Baxter equation exist, which have the form of quantum deformations around the classical R-matrix $r(u)$. Assuming that $\hbar \sim \frac{1}{u}$, equation (3.13) can be translated into an $\hbar$-independent form. The precise question then becomes whether rational solutions to the quantum Yang–Baxter equation exist which have the form

$$(\rho \otimes \rho)(\mathcal{R}(u)) = \mathbb{1} + \frac{1}{u}\,\rho(\mathrm{J}_a) \otimes \rho(\mathrm{J}_a) + \sum_{k=2}^{\infty} \frac{\mathrm{R}_k(u)}{u^k}. \tag{3.16}$$

Here we now distinguish between an abstract, more universal algebra element $\mathcal{R}$ and its representation $(\rho \otimes \rho)(\mathcal{R})$.

Before coming to the actual definition of the Yangian algebra, we have to introduce another piece of notation. Since the above operators act on tensor product states, one important question is how to generally promote representations from one site or vector space, to two or more sites. In physics language this might for instance be the question of how to go from one-particle to multi-particle representations in the context of scattering processes. The mathematical answer to this question is given by the so-called *coproduct* Δ acting for example on the elements of a Lie algebra $\mathfrak{g}$ according to $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$. In the particular case of a *Lie algebra* with generators J_a, the (primitive) coproduct is simply given by the tensor product action:

$$\Delta(\mathrm{J}_a) = \mathrm{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}_a = \mathrm{J}_{a,1} + \mathrm{J}_{a,2}. \tag{3.17}$$

In scattering processes relating asymptotic in- to out-states, on which the coproduct acts differently (see also Section 4.3), it is useful to also define an *opposite coproduct* Δ^{op} via[22]

$$\Delta^{\mathrm{op}} \equiv \mathbb{P}\, \Delta\, \mathbb{P}\,. \tag{3.18}$$

Here $\mathbb{P}$ again denotes the permutation operator that acts on the coproduct by conjugation.

Looking at (3.16), we see that at least the first order of the expansion of the rational R-matrix is completely specified by Lie algebra generators J_a. In order to define an abstract object $\mathcal{R}(u)$ that obeys the Yang–Baxter equation and has a rational form (3.16), one may thus wonder whether also the higher orders of the expansion can be defined in terms of some (possibly generalized) algebra. This is indeed the case. Inspired by Yang's first rational solution (3.11) to the quantum Yang–Baxter equation (3.5), Drinfel'd introduced the following Hopf algebra as the *Yangian* [3].

First Realization. *Given a finite-dimensional simple Lie algebra* $\mathfrak{g}$ *with generators* J_a*, the Yangian* $Y[\mathfrak{g}]$ *is defined as the algebra generated by* J_a *and* $\widehat{\mathrm{J}}_a$ *with the relations*

$$[\mathrm{J}_a, \mathrm{J}_b] = f_{abc}\mathrm{J}_c, \qquad\qquad [\mathrm{J}_a, \widehat{\mathrm{J}}_b] = f_{abc}\widehat{\mathrm{J}}_c, \tag{3.19}$$

and the following Serre relations constrain the commutator of two level-one generators[23]

$$[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \hbar^2 g_{abcdef}\{\mathrm{J}_d, \mathrm{J}_e, \mathrm{J}_f\}, \tag{3.20}$$

$$[[\widehat{\mathrm{J}}_a, \widehat{\mathrm{J}}_b], [\mathrm{J}_r, \widehat{\mathrm{J}}_s]] + [[\widehat{\mathrm{J}}_r, \widehat{\mathrm{J}}_s], [\mathrm{J}_a, \widehat{\mathrm{J}}_b]] = \hbar^2 (g_{abcdef} f_{rsc} + g_{rscdef} f_{abc})\{\mathrm{J}_d, \mathrm{J}_e, \mathrm{J}_f\}. \tag{3.21}$$

Here the f_{abc} *denote the structure constants of the algebra* $\mathfrak{g}$ *and we have*

$$g_{abcdef} = \frac{1}{24} f_{adi} f_{bej} f_{cfk} f_{ijk}, \qquad\qquad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k. \tag{3.22}$$

For completeness we already note that the Yangian defined by the above relations is a Hopf algebra (discussed in more detail below) with the coproduct[24]

$$\Delta(\mathrm{J}_a) = \mathrm{J}_a \otimes 1 + 1 \otimes \mathrm{J}_a, \qquad \Delta(\widehat{\mathrm{J}}_a) = \widehat{\mathrm{J}}_a \otimes 1 + 1 \otimes \widehat{\mathrm{J}}_a - \tfrac{1}{2}\hbar f_{abc}\mathrm{J}_b \otimes \mathrm{J}_c. \tag{3.23}$$

[22] The permutation or transposition of factors is sometimes alternatively denoted by σ acting as $\sigma \circ (a \otimes b) = b \otimes a$. That is we can alternatively write $\Delta^{\mathrm{op}} \equiv \sigma \circ \Delta$.

[23] These Serre relations are sometimes called *Drinfel'd's terrific relations* since Drinfel'd referred to the "terrific right-hand sides" of (3.20) and (3.21) in the proceedings [4]. Note that in a later version of those proceedings, the word "terrific" was exchanged for "horrible" [5]. The left hand side of (3.20) may also be written as a three-term expression of the form of the Jacobi identity, cf. [54].

[24] Here we could alternatively write $-\hbar f_{abc}\mathrm{J}_b \otimes \mathrm{J}_c = \hbar[\mathrm{J}_a \otimes 1, \mathcal{C}_\otimes]$, where $\mathcal{C}_\otimes = \mathrm{J}_a \otimes \mathrm{J}_a$ denotes the tensor Casimir operator of the underlying Lie algebra $\mathfrak{g}$.

Strictly speaking the Yangian was defined as the above algebra with $\hbar = 1$. This can usually be achieved by a rescaling of the level-one generators. Still it is elucidating to sometimes make the quantum deformation parameter $\hbar$ of this quantum group explicit.

Note that for $\mathfrak{g} = \mathfrak{sl}(2)$ the relations (3.19) imply (3.20). For $\mathfrak{g} \neq \mathfrak{sl}(2)$ (3.21) follows from (3.19) and (3.20) as already noted by Drinfel'd. Hence, we can neglect (3.21) for most cases and we will refer to (3.20) as the *Serre relations* in what follows.

As opposed to more familiar commutation relations of Lie algebras, the above definition does not specify the commutators of all generators. Rather we obtain a new generator from evaluating $\mathrm{J}_a^{(2)} \simeq f_{abc}[\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]$ in addition to $\mathrm{J}_a^{(0)} \equiv \mathrm{J}_a$ and $\mathrm{J}_a^{(1)} \equiv \widehat{\mathrm{J}}_a$. In this way one may iteratively obtain an infinite set of generators that defines the infinite dimensional Yangian algebra. The Serre relations furnish consistency conditions on this procedure as is discussed in some more detail below.

Yang–Baxter equation and Boost automorphism. Let us come back to Drinfel'd's original motivation for introducing the Yangian, namely the construction of rational solutions to the quantum Yang–Baxter equation. In order to do so, he defined the automorphism $\mathcal{B}_u$ of the Yangian algebra $Y[\mathfrak{g}]$ with the property [3]

$$\mathcal{B}_u(\mathrm{J}_a) = \mathrm{J}_a, \qquad\qquad \mathcal{B}_u(\widehat{\mathrm{J}}_a) = \widehat{\mathrm{J}}_a + u\,\mathrm{J}_a, \tag{3.24}$$

for all $u \in \mathbb{C}$. The mathematical importance of this operator is due to its role for the below construction of solutions to the Yang–Baxter equation from the Yangian. Physically, this automorphism $\mathcal{B}_u$ is realized in 1+1 dimensional models by the Lorentz boost of rapidity u. In these theories the above nontrivial action of $\mathcal{B}_u$ thus couples the internal Yangian symmetry with the spacetime symmetry. Due to this physical role we will refer to $\mathcal{B}_u$ as the *boost automorphism* in what follows.[25] Subject to the properties of this operator, the following theorem due to Drinfel'd holds.

Theorem 1 *There is a unique formal series*

$$\mathcal{R}(u) = 1 + \sum_{k=1}^{\infty} \mathcal{R}_k \frac{1}{u^k}, \qquad\qquad \mathcal{R}_k \in Y[\mathfrak{g}] \otimes Y[\mathfrak{g}]. \tag{3.25}$$

such that

$$(\Delta \otimes \mathbb{1})\mathcal{R}(u) = \mathcal{R}_{13}(u)\mathcal{R}_{23}(u), \qquad (\mathbb{1} \otimes \Delta)\mathcal{R}(u) = \mathcal{R}_{13}(u)\mathcal{R}_{12}(u), \tag{3.26}$$

and with $\Delta^{op}(a) = \mathbb{P}\,\Delta(a)\,\mathbb{P}$ *we have*

$$(\mathcal{B}_u \otimes \mathbb{1})\Delta^{op}(a) = \mathcal{R}(u)(\mathcal{B}_u \otimes \mathbb{1})\Delta(a)\mathcal{R}^{-1}(u), \tag{3.27}$$

for $a \in Y[\mathfrak{g}]$. *The operator* $\mathcal{R}(u)$ *satisfies the quantum Yang–Baxter equation. In addition, the so-called pseudo-universal R-matrix* $\mathcal{R}(u)$ *satisfies a unitarity condition of the form* $\mathcal{R}_{12}(u)\mathcal{R}_{21}(-u) = 1$ *and can be expanded around infinity in the rational form*

$$\log \mathcal{R}(u) = \frac{1}{u}\mathrm{J}_a \otimes \mathrm{J}_a + \frac{1}{u^2}(\widehat{\mathrm{J}}_a \otimes \mathrm{J}_a - \mathrm{J}_a \otimes \widehat{\mathrm{J}}_a) + \mathcal{O}\Big(\frac{1}{u^3}\Big). \tag{3.28}$$

Lastly, the R-matrix transforms under the boost automorphism as

$$(\mathcal{B}_v \otimes \mathbb{1})\mathcal{R}(u) = \mathcal{R}(u+v), \qquad (\mathbb{1} \otimes \mathcal{B}_v)\mathcal{R}(u) = \mathcal{R}(u-v). \tag{3.29}$$

[25] In the literature one also finds the names evaluation-, translation- or shift-automorphism for $\mathcal{B}_u$.

Thus for a given irreducible representation $\rho : Y[\mathfrak{g}] \to \mathrm{Mat}(n, \mathbb{C})$, the operator

$$\mathrm{R}^{\rho}(u) = (\rho \otimes \rho)(\mathcal{R}(u)) \tag{3.30}$$

is a solution to the quantum Yang–Baxter equation in the form of (3.16).[26]

Notably, the above theorem maps the search for rational solutions to the Yang–Baxter equation to the search for representations of the Yangian algebra. However, since in general we do not know the pseudo-universal R-matrix $\mathcal{R}$ (and cannot obtain it easily), this does not allow to straightforwardly construct representations of solutions to the Yang–Baxter equation. For this purpose, another theorem is very interesting [3, 54].

Theorem 2 *Given a finite dimensional irreducible representation $\rho : Y[\mathfrak{g}] \to \mathrm{End}(\mathbb{V})$, the pseudo-universal R-matrix evaluated on this representation $\mathrm{R}_{\rho}(z) = (\rho \otimes \rho)(\mathcal{R}(z))$ is the Laurent expansion about $z = \infty$ of a rational function in z. The operator*

$$\mathrm{R}(u - v) : \rho(\mathcal{B}_u(\mathbb{V})) \otimes \rho(\mathcal{B}_v(\mathbb{V})) \to \rho(\mathcal{B}_u(\mathbb{V})) \otimes \rho(\mathcal{B}_v(\mathbb{V})), \tag{3.31}$$

defined by the below constraints, is up to a scalar factor (and up to finitely many $u - v$) the same solution to the quantum Yang–Baxter equation as R_{ρ} obtained from the pseudo-universal R-matrix. The constraints on $\mathrm{R}(u, v) = \mathrm{R}(u - v)$ take the following form:

$$\textit{Level zero:} \quad (\rho \otimes \rho)\Big[\mathrm{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}_a\Big]\mathrm{R}(u, v) = \mathrm{R}(u, v)(\rho \otimes \rho)\Big[\mathrm{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}_a\Big], \tag{3.32}$$

$$\textit{Level one:} \quad (\rho \otimes \rho)\Big[(\widehat{\mathrm{J}}_a + u\mathrm{J}_a) \otimes \mathbb{1} + \mathbb{1} \otimes (\widehat{\mathrm{J}}_a + v\mathrm{J}_a) + \tfrac{1}{2} f_{abc} \mathrm{J}_b \otimes \mathrm{J}_c\Big]\mathrm{R}(u, v) =$$
$$\mathrm{R}(u, v)(\rho \otimes \rho)\Big[(\widehat{\mathrm{J}}_a + u\mathrm{J}_a) \otimes \mathbb{1} + \mathbb{1} \otimes (\widehat{\mathrm{J}}_a + v\mathrm{J}_a) - \tfrac{1}{2} f_{abc} \mathrm{J}_b \otimes \mathrm{J}_c\Big]. \tag{3.33}$$

These constraints can be evaluated as a finite system of linear equations.

Notably, all rational solutions to the quantum Yang–Baxter equation can be generated from Yangian representations in this way.

Example: $Y[\mathfrak{su}(2)]$. For illustration let us consider the rank-one example of $\mathfrak{g} = \mathfrak{su}(2)$ with representation $\rho : Y[\mathfrak{g}] \to \mathrm{Mat}(2, \mathbb{C})$ defined on one site as $\rho(\mathrm{J}_a) = \mathrm{J}_a = \frac{\sigma_a}{2i}$ and $\rho(\widehat{\mathrm{J}}_a) = 0$. Here $\sigma_{a=1,2,3}$ denotes the Pauli matrices such that $[\mathrm{J}_a, \mathrm{J}_b] = \epsilon_{abc}\mathrm{J}_c$. The above constraints at level zero, i.e.

$$[\mathrm{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}_a, \mathrm{R}(u, v)] = 0, \tag{3.34}$$

correspond to the ordinary $\mathfrak{su}(2)$ Lie algebra symmetry. For $\mathrm{R}(u, v) : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ we have only two independent irreducible representations which are mapped onto themselves by the $\mathfrak{su}(2)$ symmetry, i.e. in terms of Young tableaux:

$$\square \otimes \square = \begin{array}{|c|}\hline \\ \hline \end{array} \oplus \square\square \ . \tag{3.35}$$

In consequence there are also two $\mathfrak{su}(2)$-invariant operators of range two, e.g. the projectors onto the two irreducible representations. We already know that for $\mathfrak{u}(2)$ the tensor Casimir $\mathcal{C}_{\otimes}$ takes the form of the permutation operator $\mathbb{P}$. Obviously, this operator also commutes with the $\mathfrak{su}(2)$ symmetry. A second invariant operator is the identity $\mathbb{1}$ (the second Casimir of $\mathfrak{u}(2)$) and hence the level-zero symmetry constrains the R-matrix to be of the form

$$\mathrm{R}(u, v) = a(u, v)\ \mathbb{1} + b(u, v)\ \mathbb{P}, \tag{3.36}$$

[26] See (3.28) for the explicit application of the single-site representation of $\mathrm{J}, \widehat{\mathrm{J}}$ to the R-operator.

with arbitrary coefficients $a(u,v)$ and $b(u,v)$. After multiplication with the permutation operator $\mathbb{P}$, the level-one constraint is given by[27]

$$\Big(v\mathrm{J}_a\otimes\mathbb{1}+u\ \mathbb{1}\otimes\mathrm{J}_a-\tfrac{1}{2}\epsilon_{abc}\mathrm{J}_b\otimes\mathrm{J}_c\Big)\,\mathbb{P}\,\mathrm{R}(u,v)=\mathbb{P}\,\mathrm{R}(u,v)\Big(u\mathrm{J}_a\otimes\mathbb{1}+v\ \mathbb{1}\otimes\mathrm{J}_a-\tfrac{1}{2}\epsilon_{abc}\mathrm{J}_b\otimes\mathrm{J}_c\Big),\quad(3.37)$$

which implies

$$\tfrac{1}{2}[\epsilon_{abc}\mathrm{J}_b\otimes\mathrm{J}_c,\mathbb{P}\,\mathrm{R}(u,v)]=\Big(v\mathrm{J}_a\otimes\mathbb{1}+u\ \mathbb{1}\otimes\mathrm{J}_a\Big)\,\mathbb{P}\,\mathrm{R}(u,v)-\mathbb{P}\,\mathrm{R}(u,v)\Big(u\mathrm{J}_a\otimes\mathbb{1}+v\,\mathbb{1}\otimes\mathrm{J}_a\Big).\quad(3.38)$$

Furthermore noting that we have

$$[\epsilon_{abc}\mathrm{J}_b\otimes\mathrm{J}_c,\mathbb{P}]=\mathrm{J}_a\otimes\mathbb{1}-\mathbb{1}\otimes\mathrm{J}_a,\quad(3.39)$$

and using (3.36), we thus find

$$-\tfrac{1}{2}a(u,v)\Big(\mathrm{J}_a\otimes\mathbb{1}-\mathbb{1}\otimes\mathrm{J}_a\Big)=(u-v)b(u,v)\Big(\mathrm{J}_a\otimes\mathbb{1}-\mathbb{1}\otimes\mathrm{J}_a\Big).\quad(3.40)$$

Hence, we have $-\frac{1}{2}a(u,v)=(u-v)b(u,v)$ such that Yangian symmetry fixes the R-matrix up to an overall factor to be of Yang's form (3.12):

$$\mathrm{R}(u,v)=a(u,v)\Big(\mathbb{1}-\frac{1}{2(u-v)}\ \mathbb{P}\Big).\quad(3.41)$$

Note that for the above normalization of a basis of $\mathfrak{u}(2)$ we have $\mathcal{C}_\otimes=\mathbb{P}$. This example for $\mathfrak{g}=\mathfrak{su}(2)$ illustrates the basic principle of how to fix the matrix structure of an R- or S-matrix from Yangian symmetry and can be generalized to more complicated algebras $\mathfrak{g}$.

Representations and Serre relations. If you encounter a symmetry in a physical model that has generators J_a and $\widehat{\mathrm{J}}_a$ and follows the coproduct structure (3.23), this is a promising sign that you are dealing with a Yangian algebra. However, you will have to verify that your generators obey the Serre relations, which is typically hard work. It may thus be useful to understand the nature of these Serre relations a bit better.

The above coproduct is an algebra homomorphism, that is the following relation should hold for $a,b\in Y[\mathfrak{g}]$:

$$\Delta([a,b])=[\Delta(a),\Delta(b)].\quad(3.42)$$

This homomorphism property is trivially obeyed for some commutators of generators with the coproduct structure (3.23), i.e. one easily verifies that

$$\Delta([\mathrm{J}_a,\mathrm{J}_b])=[\Delta(\mathrm{J}_a),\Delta(\mathrm{J}_b)],\qquad\Delta([\mathrm{J}_a,\widehat{\mathrm{J}}_b])=[\Delta(\mathrm{J}_a),\Delta(\widehat{\mathrm{J}}_b)].\quad(3.43)$$

Consider for instance the second case, whose left- and right hand sides explicitly evaluate to

$$\Delta([\mathrm{J}_a,\widehat{\mathrm{J}}_b])=\Delta(f_{abc}\widehat{\mathrm{J}}_c)=f_{abc}\,\mathbb{1}\otimes\widehat{\mathrm{J}}_c+f_{abc}\widehat{\mathrm{J}}_c\otimes\mathbb{1}-\tfrac{1}{2}f_{abc}f_{cde}\mathrm{J}_d\otimes\mathrm{J}_e,\quad(3.44)$$

$$\begin{aligned}[\Delta(\mathrm{J}_a),\Delta(\widehat{\mathrm{J}}_b)]&=[\mathbb{1}\otimes\mathrm{J}_a+\mathrm{J}_a\otimes\mathbb{1},\mathbb{1}\otimes\widehat{\mathrm{J}}_b+\widehat{\mathrm{J}}_b\otimes\mathbb{1}-\tfrac{1}{2}f_{bcd}\mathrm{J}_c\otimes\mathrm{J}_d]\\&=f_{abc}\,\mathbb{1}\otimes\widehat{\mathrm{J}}_c+f_{abc}\widehat{\mathrm{J}}_c\otimes\mathbb{1}-\tfrac{1}{2}(f_{bdc}f_{ace}+f_{bce}f_{acd})\mathrm{J}_d\otimes\mathrm{J}_e.\end{aligned}\quad(3.45)$$

Both sides are equal upon using the Jacobi identity $f_{abc}f_{cde}+f_{dac}f_{cbe}+f_{bdc}f_{cae}=0$. On the other hand, the relation

$$\Delta([\widehat{\mathrm{J}}_a,\widehat{\mathrm{J}}_b])=[\Delta(\widehat{\mathrm{J}}_a),\Delta(\widehat{\mathrm{J}}_b)]\quad(3.46)$$

[27]Sometimes one introduces $\check{\mathrm{R}}=\mathbb{P}\,\mathrm{R}$ and rephrases the above statements in terms of this operator.

or when making the representation ρ explicit

$$\rho(\Delta([\widehat{\mathrm{J}}_a, \widehat{\mathrm{J}}_b)]) = [\rho(\Delta(\widehat{\mathrm{J}}_a)), \rho(\Delta(\widehat{\mathrm{J}}_b))], \tag{3.47}$$

does not trivially follow from the definition of the coproduct, but it implies non-trivial constraints on the representation ρ of the Yangian generators. This can be seen by noting that the left hand side of (3.47) forms part of the antisymmetrized tensor product of the adjoint representation with itself

$$(\mathrm{adj} \otimes \mathrm{adj})_{\mathrm{asym}} = \mathrm{adj} \oplus \mathbb{X}. \tag{3.48}$$

This relation defines the representation $\mathbb{X}$ that does typically not contain the adjoint representation. The adjoint part defines the coproduct for the level-two Yangian generators while the Serre relations furnish a sufficient criterion for the vanishing of the $\mathbb{X}$-component, cf. e.g. [55] for more details. In fact, if the Serre relations are satisfied for the one-site representation, they will also hold for the n-site representation since the coproduct preserves the Serre relations.

Construction of representations. As pointed out by Drinfel'd, given a Lie algebra represenation ρ one may choose the following one-site representation ρ_0 of the Yangian generators

$$\rho_0(\mathrm{J}_a) = \rho(\mathrm{J}_a), \qquad\qquad \rho_0(\widehat{\mathrm{J}}_a) = 0. \tag{3.49}$$

The left hand side of (3.20) vanishes in this case. In order to show that our representation ρ_0 obeys the Serre relations, we have to show that the $\mathbb{X}$-projection of the right hand side of (3.20) vanishes for the one-site representation:

$$\rho_0(\{\mathrm{J}_a, \mathrm{J}_b, \mathrm{J}_c\})|_{\mathbb{X}} = 0. \tag{3.50}$$

In [3] Drinfel'd indicated the existence of such representations for all types of algebras $\mathfrak{g}$ except for $\mathfrak{e}_8$.[28] Once (3.50) is shown for the one-site representation, one promotes the representation to multiple sites via the coproduct which preserves the Serre relations.

Evaluation representation. For some representations ρ of the Lie algebra $\mathfrak{g}$ there exists a so-called *evaluation representation* ρ_u of the Yangian algebra given by

$$\rho_u(\mathrm{J}_a) = \rho_0(\mathrm{J}_a), \qquad\qquad \rho_u(\widehat{\mathrm{J}}_a) = u\,\rho_0(\mathrm{J}_a). \tag{3.51}$$

As discussed above for $u = 0$, this choice puts the constraint on the representation that the right hand side of the Serre relations vanishes for the one-site representation since the left hand side is trivially zero. The evaluation representation can also be defined using the boost automorphism and the above representation ρ_0 as follows:

$$\rho_u(\mathrm{J}_a) = \rho_0(\mathrm{J}_a), \qquad\qquad \rho_u(\widehat{\mathrm{J}}_a) = \rho_0\Big(\mathcal{B}_u(\widehat{\mathrm{J}}_a)\Big). \tag{3.52}$$

This representation is important for evaluating the Yangian symmetry of the two-particle S-matrix $\mathrm{S}(u, v)$ where u and v represent particle rapidities, cf. Section 4.3. For this purpose it will be useful to explicitly evaluate the two-site representation using the coproduct and the two-site boost automorphism

$$\mathcal{B}_u \otimes \mathcal{B}_v(\Delta(\widehat{\mathrm{J}}_a)) = u\,\mathrm{J}_a \otimes \mathbb{1} + v\;\mathbb{1} \otimes \mathrm{J}_a + \widehat{\mathrm{J}}_a \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathrm{J}}_a - \tfrac{1}{2} f_{abc} \mathrm{J}_b \otimes \mathrm{J}_c, \tag{3.53}$$

[28] These representations also play an important role in the AdS/CFT correspondence. The Serre relations were shown for representations of $\mathfrak{psu}(2,2|4)$ [21] and $\mathfrak{osp}(4|6)$ [55] that realize the Yangian symmetry in $\mathcal{N} = 4$ super Yang–Mills and $\mathcal{N} = 6$ superconformal Chern–Simons theory.

	Classical Mechanics	**Quantum Mechanics**
States	Points on manifold M	1d subspaces of Hilbert space H
Observables	Algebra of functions on M	Algebra of operators on H
	Classical Group	**Quantum Group**
States	Elements of group G	?
Observables	Algebra of functions on G	?

Quantization

Commutative → Non-commutative

Table 1: What is a quantum group?

which yields[29]

$$\rho_u \otimes \rho_v\Big(\Delta(\widehat{\mathrm{J}}_a)\Big) = \rho_0 \otimes \rho_0\Big(\mathcal{B}_u \otimes \mathcal{B}_v(\Delta(\widehat{\mathrm{J}}_a))\Big) = \rho \otimes \rho\Big(u\,\mathrm{J}_a \otimes \mathbb{1} + v\;\mathbb{1} \otimes \mathrm{J}_a - \tfrac{1}{2} f_{abc}\mathrm{J}_b \otimes \mathrm{J}_c\Big). \tag{3.54}$$

3.2 The Yangian as a Hopf Algebra and Quantum Group.

We continue our study of the mathematical structure behind the Yangian together with Drinfel'd:

> *"Now let us consider the elements of a group* G *as states and functions on* G *as observables. The notion of group is usually defined in terms of states. To quantize it one has to translate it first into the language of observables. This translation is well known, but let us recall it nevertheless."* V. Drinfel'd 1986 [5].

Hopf algebras. We are now interested in generalizing the quantization of classical mechanics to the case of groups or algebras, cf. Table 1. Following [4, 5], we thus want to understand how the properties of a group considered as the space of states, translate into the language of observables. For this purpose we remember that a group is defined as a pair (G, f) of a set G and a group operation f such that

$$f : \mathrm{G} \times \mathrm{G} \to \mathrm{G}. \tag{3.55}$$

Remember also that a group is defined to be associative which can be conveniently displayed using the following diagram:

$$f(f \times \mathbb{1})(x,y,z) = f(\mathbb{1} \times f)(x,y,z): \qquad \begin{array}{ccccc} & & \mathrm{G}\times\mathrm{G} & & \\ & \overset{f\times\mathbb{1}}{\nearrow} & & \overset{f}{\searrow} & \\ \mathrm{G}\times\mathrm{G}\times\mathrm{G} & & & & \mathrm{G}\,, \\ & \underset{\mathbb{1}\times f}{\searrow} & & \underset{f}{\nearrow} & \\ & & \mathrm{G}\times\mathrm{G} & & \end{array} \tag{3.56}$$

for $x, y, z \in \mathrm{G}$. We consider the algebra $\mathfrak{a} = \mathrm{Fun}(\mathrm{G})$ (the observables) consisting of functions on the group G. The group map f induces an algebra homomorphism[30] which is dubbed *coproduct*

[29] The evaluation representation may be used to define Yangian-invariant deformations of scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills [56] and $\mathcal{N} = 6$ superconformal Chern–Simons theory [57].

[30] An algebra homomorphism between two algebras $\mathfrak{a}$ and $\mathfrak{b}$ over the field $\mathbb{C}$ is a map $\Delta : \mathfrak{a} \to \mathfrak{b}$ such that for all $k \in \mathbb{C}$ and $a, b \in \mathfrak{a}$:

$$\Delta(ka) = k\Delta(a), \qquad \Delta(a+b) = \Delta(a) + \Delta(b), \qquad \Delta(ab) = \Delta(a)\Delta(b). \tag{3.57}$$

or *comultiplication*[31]

$$\Delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}, \tag{3.58}$$

where $\mathfrak{a} \otimes \mathfrak{a} = \mathrm{Fun}(\mathrm{G} \times \mathrm{G})$. This is the coproduct we already encounterd above. As mentioned, in physics the coproduct furnishes a prescription for how to extend the symmetry from one- to multi-particle states in a fashion that is compatible with the underlying algebraic structure.

Translating the associativity of the group map to the coproduct Δ we find the property of *coassociativity*, i.e.

$$(\mathbb{1} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathbb{1})\Delta(a) : \qquad \begin{array}{ccccc} & & \mathfrak{a} \otimes \mathfrak{a} & & \\ & \overset{\Delta}{\nearrow} & & \overset{\mathbb{1}\otimes\Delta}{\searrow} & \\ \mathfrak{a} & & & & \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a} \\ & \underset{\Delta}{\searrow} & & \underset{\Delta\otimes\mathbb{1}}{\nearrow} & \\ & & \mathfrak{a} \otimes \mathfrak{a} & & \end{array} . \tag{3.59}$$

Finally we need to find the analogue of the group inversion $x \mapsto x^{-1}$, which is denoted the *antipode*

$$s : \mathfrak{a} \to \mathfrak{a}, \tag{3.60}$$

and the analogue of the unit element e of the group, which is denoted the *counit*:

$$\epsilon : \mathfrak{a} \to \mathbb{C}. \tag{3.61}$$

We also have an ordinary multiplication $m : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$, and the unit map $\eta : \mathbb{C} \to \mathfrak{a}$ defined as $\eta : c \mapsto c \cdot \mathbb{1}$, for $c \in \mathbb{C}$ and $\mathbb{1} \in \mathfrak{a}$. These maps should obey the following commutative diagrams which correspond to $e \cdot x = x$ and $x \cdot e = x$ for $x \in \mathrm{G}$:

$$a = (\mathbb{1} \otimes \epsilon)\Delta(a) : \quad \begin{array}{ccc} \mathfrak{a} & \xrightarrow{\mathbb{1}} & \mathfrak{a} \\ {\scriptstyle\Delta}\downarrow & & \| \\ \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{\mathbb{1}\otimes\epsilon} & \mathfrak{a} \otimes \mathbb{C} \end{array} , \qquad a = (\epsilon \otimes \mathbb{1})\Delta(a) : \quad \begin{array}{ccc} \mathfrak{a} & \xrightarrow{\mathbb{1}} & \mathfrak{a} \\ {\scriptstyle\Delta}\downarrow & & \| \\ \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{\epsilon\otimes\mathbb{1}} & \mathbb{C} \otimes \mathfrak{a} \end{array} .$$

Lastly, all of the introduced maps should be compatible with each other and obey the relations

$$m(s \otimes \mathbb{1})\Delta(a) = \eta(\epsilon(a)) = m(\mathbb{1} \otimes s)\Delta(a) : \qquad \begin{array}{ccccc} & & \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{s\otimes\mathbb{1}} & \mathfrak{a} \otimes \mathfrak{a} \\ & \overset{\Delta}{\nearrow} & & & \searrow^{m} \\ \mathfrak{a} & & \xrightarrow{\epsilon} & \mathbb{C} \xrightarrow{\eta} & \mathfrak{a} \\ & \underset{\Delta}{\searrow} & & & \nearrow_{m} \\ & & \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{\mathbb{1}\otimes s} & \mathfrak{a} \otimes \mathfrak{a} \end{array} . \tag{3.62}$$

The above properties of $(\mathfrak{a}, \Delta, s, \epsilon, m, \eta)$ define the commutative *Hopf algebra* $\mathfrak{a}$. This Hopf algebra furnishes a notion of the class of observables in the context of groups.

Importantly, a Hopf algebra is called *cocommutative* if the opposite coproduct obeys

$$\Delta^{\mathrm{op}}(a) = \Delta(a), \tag{3.63}$$

[31]The coproduct is induced via $(\Delta(a))(xy) := a(xy)$ for $a \in \mathfrak{a}$ and $x, y \in \mathrm{G}$, cf. e.g. [58] for more details.

for $a \in \mathfrak{a}$. Remember that the opposite coproduct was defined as $\Delta^{\rm op} = \mathbb{P}\,\Delta\,\mathbb{P}$. The analogy between the group product f and the coproduct Δ is underlined by the fact that the above Hopf algebra $\mathfrak{a}$ is cocommutative iff the group G is abelian. In the above spirit, we can thus understand the quantization of a Hopf algebra as the replacement of a cocommutative coproduct by a non-cocommutative coproduct. This is in close analogy to the transition from classical to quantum mechanics, where a commutative product is replaced by a non-commutative one, e.g. $[x, p] = 0$ goes to $[\hat{x}, \hat{p}] = i\hbar$.

Example: The universal enveloping algebra $U[\mathfrak{g}]$. Given a Lie algebra $\mathfrak{g}$, one can define the *universal enveloping algebra (UEA)* $\mathfrak{a} = U[\mathfrak{g}]$ as the quotient of the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes^n} = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots \tag{3.64}$$

by the elements $a \otimes b - b \otimes a - [a, b]$ for $a, b \in \mathfrak{g}$. Here $\mathbb{K}$ is the field associated to the Lie algebra $\mathfrak{g}$. Hence, the UEA may be considered as the space of polynomials of elements of $\mathfrak{g}$ modulo the commutator, i.e. for our Lie algebra generators the combination $\mathrm{J}_a \otimes \mathrm{J}_b - \mathrm{J}_b \otimes \mathrm{J}_a$ is identified with $f_{abc}\mathrm{J}_c$. The UEA can be equipped with a Hopf algebra structure by defining

$$\Delta(a) = a \otimes 1 + 1 \otimes a, \qquad s(a) = -a, \qquad \epsilon(a) = 0, \tag{3.65}$$

for $a \in \mathfrak{g}$ which naturally extends to $U[\mathfrak{g}]$. Note that this coproduct is cocommutative, i.e. invariant under exchanging the factors on the left and right hand side of $\otimes$, which is consistent with our idea of an un-quantized, i.e. classical algebra.

Quantum groups. The term *quantum group* introduced by Drinfel'd is generically employed to refer to a deformed algebraic structure. This deformation is typically parametrized by a deformation parameter which we call $\hbar$ to remind of the physical quantum deformation of classical mechanics: $[x, p] = 0 \to [\hat{x}, \hat{p}] = i\hbar$. In particular, the name quantum group does often not refer to a group in the ordinary mathematical sense. Here we will understand quantum groups as special examples of Hopf algebras,[32] namely quantizations $U_\hbar[\mathfrak{g}]$ of the universal enveloping algebra $U[\mathfrak{g}]$ of an underlying algebra $\mathfrak{g}$. In accordance with the relation between classical and quantum mechanics, this quantization goes along with replacing a cocommutative coproduct by a non-cocommutative one.

Example: $U_\hbar[\mathfrak{sl}(2)]$. Consider the example of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ (cf. e.g. [54]) with generators X^+, X^- and H obeying

$$[X^+, X^-] = H, \qquad [H, X^\pm] = \pm 2X^\pm. \tag{3.66}$$

Based on this algebra, we may define the universal enveloping algebra $U[\mathfrak{sl}(2)]$ as introduced above. Then the primitive coproduct

$$\Delta(X^+) = X^+ \otimes \mathbb{1} + \mathbb{1} \otimes X^+, \quad \Delta(X^-) = X^- \otimes \mathbb{1} + \mathbb{1} \otimes X^-, \quad \Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H, \tag{3.67}$$

may be understood as an algebra homomorphism on the universal enveloping algebra $U[\mathfrak{sl}(2)]$. A deformation $U_\hbar[\mathfrak{sl}(2)]$ of the universal enveloping algebra is induced by deforming the above commutation relations or structure constants, respectively, to

$$[X^+, X^-] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^\hbar - e^{-\hbar}}, \qquad [H, X^\pm] = \pm 2X^\pm. \tag{3.68}$$

[32] Strictly speaking quantum groups are rather dual (but equivalent) to Hopf algebras, but this distinction is often not made. See for instance [59] for some explicit discussions.

Note that in the classical limit $\hbar \to 0$ we obtain the undeformed algebra (3.66):

$$\lim_{\hbar\to 0} \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = H. \tag{3.69}$$

The non-cocommutative coproduct for the three types of generators takes the form

$$\Delta(X^+) = X^+ \otimes e^{\hbar H} + \mathbb{1} \otimes X^+, \qquad \Delta(X^-) = X^- \otimes \mathbb{1} + e^{-\hbar H} \otimes X^-, \tag{3.70}$$

and

$$\Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H. \tag{3.71}$$

For $\hbar \to 0$ the coproduct is the primitive one (3.67), which is cocommutative.

Quasitriangular Hopf algebras and universal R-matrix. Let us briefly introduce some further important concepts related to integrable models and the Yangian. A Hopf algebra $\mathfrak{a}$ is called *almost cocommutative*, if an element $\mathcal{R} \in \mathfrak{a} \otimes \mathfrak{a}$ exists such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{3.72}$$

for all $a \in \mathfrak{a}$, where $\Delta^{\mathrm{op}} = \mathbb{P}\,\Delta\,\mathbb{P}$. That is if the opposite coproduct Δ^{op} and the coproduct Δ are similar. Comparing (3.72) to (3.27) we see that this is not the case for the Yangian (see below paragraph). An almost cocommutative Hopf algebra $(\mathfrak{a}, \mathcal{R})$ is called *quasitriangular* if

$$(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathbb{1} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.73}$$

If $\mathfrak{a}$ is quasitriangular, the element $\mathcal{R}$ is called the *universal R-matrix* of $(\mathfrak{a}, \mathcal{R})$. The universal R-matrix of a quasi-triangular Hopf algebra satisfies the quantum Yang–Baxter equation as well as the relation

$$(s \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathbb{1} \otimes s^{-1})(\mathcal{R}), \tag{3.74}$$

where s denotes the antipode. The property (3.74) is important for physical applications since it represents the *crossing* relation when $\mathcal{R}$ is given by a scattering matrix with Hopf algebra symmetry, cf. e.g. [28]. For completeness let us mention that a quasi-triangular Hopf algebra is called *triangular* if $\mathcal{R}_{12}\mathcal{R}_{21} = \mathbb{1}$.

The Yangian as a Hopf algebra and quantum group. The Yangian defined above is a Hopf algebra with the coproduct (we may set $\hbar = 1$)

$$\Delta(\mathrm{J}_a) = \mathrm{J}_a \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}_a, \qquad \Delta(\widehat{\mathrm{J}}_a) = \widehat{\mathrm{J}}_a \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathrm{J}}_a - \tfrac{1}{2}\hbar f_{abc}\mathrm{J}_b \otimes \mathrm{J}_c. \tag{3.75}$$

The antipode acts on the generators according to

$$s(\mathrm{J}_a) = -\mathrm{J}_a, \qquad s(\widehat{\mathrm{J}}_a) = -\widehat{\mathrm{J}}_a + \tfrac{1}{2}\hbar f_{abc}\mathrm{J}_b\mathrm{J}_c, \tag{3.76}$$

and the counit acts trivially as[33]

$$\epsilon(\mathrm{J}_a) = 0, \qquad \epsilon(\widehat{\mathrm{J}}_a) = 0. \tag{3.77}$$

The Yangian is *not* quasitriangular since the *pseudo-universal* operator $\mathcal{R}$ of the Yangian is not an element of $Y[\mathfrak{g}] \otimes Y[\mathfrak{g}]$. This requires the introduction of the above boost automorphism

[33]For simple Lie algebras $\mathfrak{g}$ one can rewrite $f_{abc}\mathrm{J}_b\mathrm{J}_c = \frac{1}{2}\mathcal{C}\mathrm{J}_a$ with $\mathcal{C}$ being the quadratic Casimir of $\mathfrak{g}$ in the adjoint representation.

(3.24), and (3.27) represents the pseudo-triangularity condition analogous to (3.72) for the quasitriangular case. Alternatively one could consider the so-called *Yangian double* which possesses a universal R-matrix, see e.g. [54].

On the level of the abstract algebra, the evaluation representation (3.51) discussed above may be induced by an *evaluation homomorphism* from the Yangian to the universal enveloping algebra

$$\mathrm{ev}_u : Y[\mathfrak{g}] \to U[\mathfrak{g}], \qquad \mathrm{ev}_u(\mathrm{J}_a) = \mathrm{J}_a, \qquad \mathrm{ev}_u(\widehat{\mathrm{J}}_a) = u\,\mathrm{J}_a. \tag{3.78}$$

This homomorphism, however, turns out to exist only for $\mathfrak{g} = \mathfrak{sl}(2)$, while it takes a more complicated form for $\mathfrak{sl}(N)$ with $N > 2$ and does not exist for symmetry algebras of type different from $\mathfrak{a}_N$ (in the Dynkin classification of simple Lie groups) [54].

The Yangian is a quantum deformation of what? The Yangian is a deformation of the UEA of the so-called *polynomial algebra* $\mathfrak{g}[u]$. Given a Lie algebra $\mathfrak{g}$, the polynomial algebra $\mathfrak{g}[u]$ is defined as the space of polynomials in u with values in $\mathfrak{g}$. This means that $\mathfrak{g}[u]$ is spanned by monomials of the form $\mathrm{J}^a_n = u^n \mathrm{J}^a$ with $n = 0, \ldots, \infty$. The simplest way to construct representations of the polynomial algebra is via the evaluation homomorphism

$$\mathrm{ev}_u : \mathfrak{g}[u] \to \mathfrak{g}, \tag{3.79}$$

which evaluates a polynomial at a fixed point $u \in \mathbb{C}$. The evaluation homomorphism of the Yangian algebra discussed above represents the quantum generalization of this map. Taking $\hbar \to 0$ in the defining relations of the Yangian, one obtains the UEA of $\mathfrak{g}[u]$ with the correct Hopf structure (3.65).

> *"By the way, we are lucky that $Y[\mathfrak{g}]$ is pseudo-triangular and not triangular: otherwise $Y[\mathfrak{g}]$ would be isomorphic (as an algebra) to a universal enveloping algebra and life would be dull."* V. Drinfel'd 1986 [5]

3.3 Second and Third Realization

While these lectures put more weight on the original, first realization and its connection to physical systems, it should be emphasized, that further notable realizations of the Yangian algebra exist and were discussed by Drinfel'd. In the context of physical systems, in particular the third, so-called RTT realization establishes a connection to earlier work on integrability and the quantum inverse scattering method, see also Section 5.

3.3.1 Second Realization

In 1988 Drinfel'd introduced a new realization of the Yangian that will be briefly discussed in this subsection. Drinfel'd's motivation for studying this new realization was some shortcomings of the first realization:

> *"Unfortunately, the realization given in [7] and [5] of Yangians and quantized affine algebras*[34] *is not suitable for the study of finite-dimensional representations of these algebras."* V. Drinfel'd 1988 [6]

[34] Here Drinfel'd refers to the quantum algebras that take a similar role for trigonometric solutions to the Yang–Baxter equation as the Yangian for rational solutions.

The new realization of the Yangian given below can be used to demonstrate a one-to-one correspondence between irreducible finite-dimensional representations of the Yangian and sets of polynomials [6]. While this correspondence proves useful for studying Yangian representations, it is beyond the scope of these lectures.

The second realization is particularly interesting since it specifies the defining relations for all generators as opposed to the first realization. This is important for the construction of a universal R-matrix of the so-called Yangian double.

In order to understand the approach towards this new realization of the Yangian, let us first get some inspiration from ordinary Lie algebras.

Semisimple Lie algebras. Due to Serre, every finite dimensional semisimple Lie algebra can be represented in terms of a Chevalley basis of generators. More explicitly, an $n \times n$ Cartan matrix $A = (a_{ij})$ and a set of $3n$ generators $\{X_i^\pm, H_i\}_{i=1}^n$ which satisfy the Serre relations, uniquely define a semisimple Lie algebra $\mathfrak{g}$ of rank n. The generators obey the commutation relations (here $[\,\cdot\,,\,\cdot\,]$ denotes the Lie bracket)

$$[H_i, H_j] = 0, \qquad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \qquad [X_i^+, X_j^-] = \delta_{ij} H_j, \tag{3.80}$$

as well as the Serre relations

$$i \neq j: \qquad \mathrm{ad}(X_i^\pm)^{1-a_{ij}}(X_j^\pm) = [X_i^\pm, [X_i^\pm, \dots [X_i^\pm, X_j^\pm]]] = 0. \tag{3.81}$$

Note that the H_i generate a Cartan subalgebra of $\mathfrak{g}$. The simplest example with $n = 1$ is a one-dimensional Cartan matrix (element) $A = a_{11} = 2$ such that the above relations yield the well known commutation relations of $\mathfrak{sl}(2)$:

$$[H, X^\pm] = \pm 2 X^\pm, \qquad [X^+, X^-] = H. \tag{3.82}$$

One example for infinite dimensional generalizations of semisimple Lie algebras generated in this way are the Kac–Moody algebras. Another example can be defined as follows.

Chevalley–Serre realization of the Yangian. In [6] Drinfeld introduced a second realization of the Yangian that follows the above Chevalley–Serre pattern.

Second Realization. *The algebra $\mathfrak{c}$ defined in the following way is isomorphic to $Y[\mathfrak{g}]$. Given a simple Lie algebra $\mathfrak{g}$ with inner product $(\,\cdot\,,\,\cdot\,)$, the associative algebra $\mathfrak{c}$ with generators $x_{ik}^\pm$ and h_{ik} is defined by the relations (here $[\,\cdot\,,\,\cdot\,]$ denotes the commutator in $\mathfrak{c}$)*

$$[h_{ik}, h_{jl}] = 0, \qquad [h_{i0}, x_{jl}^\pm] = \pm a_{ij} x_{jl}^\pm, \qquad [x_{ik}^+, x_{jl}^-] = \delta_{ij} h_{i,k+l} \tag{3.83}$$

and

$$[h_{i,k+1}, x_{jl}^\pm] - [h_{ik}, x_{j,l+1}^\pm] = \pm \tfrac{1}{2} a_{ij} (h_{ik} x_{jl}^\pm + x_{jl}^\pm h_{ik}), \tag{3.84}$$

$$[x_{i,k+1}^\pm, x_{jl}^\pm] - [x_{ik}^\pm, x_{j,l+1}^\pm] = \pm \tfrac{1}{2} a_{ij} (x_{ik}^\pm x_{jl}^\pm + x_{jl}^\pm x_{ik}^\pm), \tag{3.85}$$

as well as

$$i \neq j,\ m = 1 - a_{ij} \quad \Rightarrow \quad \mathrm{Sym}_{\{k\}} [x_{ik_1}^\pm, [x_{ik_2}^\pm, \dots [x_{ik_m}^\pm, x_{jl}^\pm]] \dots] = 0. \tag{3.86}$$

Here $A = (a_{ij})$ denotes the Cartan matrix of $\mathfrak{g}$. The indices i, j run over $1, \dots, \mathrm{rank}(\mathfrak{g})$ and $k, l = 1, 2, \dots$. Furthermore $\mathrm{Sym}_{\{k\}}$ denotes symmetrization in $k_1, \dots, k_m$ with weight 1.

Let $\{H_i, X_i^\pm\}$ denote a Chevalley–Serre basis of the Lie algebra $\mathfrak{g}$ with $\widehat{H}_i$ and $\widehat{X}_i^\pm$ representing the level-one generators introduced in the context of the first realization. Then Drinfel'd's isomorphism φ between the Yangian and the algebra $\mathfrak{c}$ takes the form

$$\varphi(H_i) = h_{i0}, \qquad \varphi(X_i^+) = x_{i0}^+, \qquad \varphi(X_i^-) = x_{i0}^-, \tag{3.87}$$

$$\varphi(\widehat{H}_i) = h_{i1} + \varphi(v_i), \qquad \varphi(\widehat{X}_i^+) = x_{i1}^+ + \varphi(w_i), \qquad \varphi(\widehat{X}_i^-) = x_{i1}^- + \varphi(z_i), \tag{3.88}$$

with

$$v_i = +\tfrac{1}{4}\sum_\alpha (\alpha,\alpha_i)(e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha) - \tfrac{1}{2}H_i^2, \tag{3.89}$$

$$w_i = +\tfrac{1}{4}\sum_\alpha ([X_i^+, e_\alpha]e_{-\alpha} + e_{-\alpha}[X_i^+, e_\alpha]) - \tfrac{1}{4}(X_i^+ H_i + H_i X_i^+), \tag{3.90}$$

$$z_i = -\tfrac{1}{4}\sum_\alpha ([X_i^-, e_{-\alpha}]e_\alpha + e_\alpha[X_i^-, e_{-\alpha}]) - \tfrac{1}{4}(X_i^- H_i + H_i X_i^-). \tag{3.91}$$

Here α runs over all positive roots and the e's denote the generators of the Cartan-Weyl basis.

Drinfel'd furthermore noted that if the right hand side of (3.84) and (3.85) is set to zero (which corresponds to the classical limit), then the algebra $\mathfrak{c}$ is isomorphic to the universal enveloping algebra $U[\mathfrak{g}[u]]$ with an isomorphism of the structure

$$h_{ik} \mapsto H_i u^k, \qquad x_{ik}^+ \mapsto X_i^+ u^k, \qquad x_{ik}^- \mapsto X_i^- u^k. \tag{3.92}$$

An explicit expression for the coproduct of the second realization is not known. The boost autmorphism in this realization is given by [54]

$$\mathcal{B}_u(h_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} u^{r-s}\, h_{i,s}, \qquad \mathcal{B}_u(x_{i,r}^\pm) = \sum_{s=0}^{r} \binom{r}{s} u^{r-s}\, x_{i,s}^\pm. \tag{3.93}$$

Notably, one may modify the above definition of the Yangian employing only a finite number of the generators of the second realization, a result due to Levendorskiĭ [60].

3.3.2 Third Realization

We will now consider a third realization of the Yangian that was implicitly studied by the Leningrad school [61] before Drinfel'd's seminal papers. Drinfel'd himself made the connection to the earlier work explicit:

> *"Finally, I am going to mention a realization of $Y[\mathfrak{g}]$ which is often useful and which appeared much earlier than the general definition of $Y[\mathfrak{g}]$."* V. Drinfel'd 1986 [5]

In particular, this realization establishes the connection to the fundamental equations underlying the algebraic Bethe Ansatz [29].

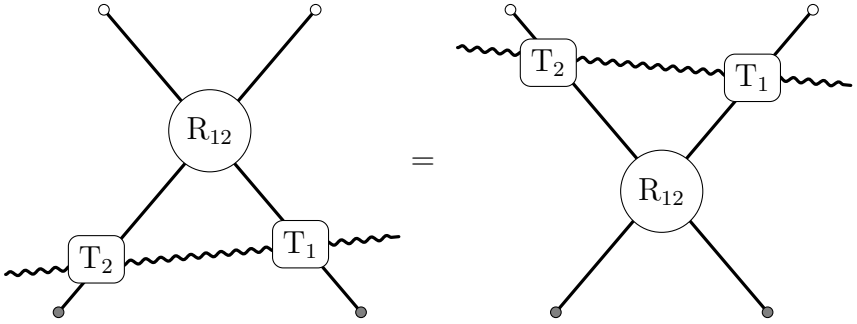


Figure 3: Illustration of the RTT-relations.

RTT Realization. *Fix a nontrivial irreducible representation of the Yangian* $\rho : Y[\mathfrak{g}] \to \mathrm{Mat}(n, \mathbb{C})$. *Let furthermore* $\mathrm{R}(u) = (\rho \otimes \rho)(\mathcal{R}(u))$, *where* $\mathcal{R}(u)$ *denotes the rational solution to the Yang–Baxter equation given in (3.25). Define a Hopf algebra* $\mathfrak{a}_\rho$ *by the RTT-relations*

$$\mathrm{R}_{12}(u-v)\mathrm{T}_1(u)\mathrm{T}_2(v) = \mathrm{T}_2(v)\mathrm{T}_1(u)\mathrm{R}_{12}(u-v). \tag{3.94}$$

Here, denoting the $n \times n$ *identity matrix by* $\mathbb{1}$, *we have*

$$\mathrm{T}_1(u) = \mathrm{T}(u) \otimes \mathbb{1}, \qquad \mathrm{T}_2(v) = \mathbb{1} \otimes \mathrm{T}(v), \tag{3.95}$$

and with $\mathrm{T}(u) = \sum_{\alpha\beta} t^\alpha{}_\beta(u) E_\alpha{}^\beta$ *and* $1 \leq \alpha, \beta \leq N$ *the Laurent expansion of the matrix elements* $t^\alpha{}_\beta(u)$ *takes the form*

$$t^\alpha{}_\beta(u) = \delta^\alpha_\beta + \sum_{k=1}^{\infty} \frac{(t^{(k)})^\alpha{}_\beta}{u^k}. \tag{3.96}$$

The Hopf algebra $\mathfrak{a}_\rho$ *is generated by the operators* $(t^{(k)})^\alpha{}_\beta$ *with* $k = 1, 2, \ldots$ *and the coproduct*

$$\Delta(t^\alpha{}_\beta(u)) = \sum_\gamma t^\alpha{}_\gamma(u) \otimes t^\gamma{}_\beta(u). \tag{3.97}$$

One has an epimorphism (surjection) $\mathfrak{a}_\rho \to Y[\mathfrak{g}]$ *defined by* $T(u) \mapsto (\mathbb{1} \otimes \rho)(\mathcal{R}(u))$, *where the expansion of* $\mathcal{R}$ *in terms of the generators in the first realization was given by (3.28):*

$$\log \mathcal{R}(u) = \frac{1}{u}\mathrm{J}_a \otimes \mathrm{J}_a + \frac{1}{u^2}(\widehat{\mathrm{J}}_a \otimes \mathrm{J}_a - \mathrm{J}_a \otimes \widehat{\mathrm{J}}_a) + \mathcal{O}\Big(\frac{1}{u^3}\Big). \tag{3.98}$$

To obtain $Y[\mathfrak{g}]$ *from* $\mathfrak{a}_\rho$ *(i.e. to define a bijection), one generically has to add an auxiliary relation of the form*

$$c(u) = 1, \tag{3.99}$$

where $\Delta(c(u)) = c(u) \otimes c(u)$ *and such that* $[a, c(u)] = 0$, *for all* $a \in \mathfrak{a}_\rho$.

Note that this realization makes explicit reference to a representation ρ from the start as opposed to the previous two realizations. The above RTT-relations (3.94) follow from the Yang–Baxter equation (3.5) if we identify $\mathrm{R}_{i3}(u)$ with $\mathrm{T}_i(u) \equiv \mathrm{T}_{i3}(u)$ and set $u \equiv u_1$, $v \equiv u_2$ and $u_3 \equiv 0$. The RTT-relations relate the products $\mathrm{T}_1(u)\mathrm{T}_2(v)$ and $\mathrm{T}_2(v)\mathrm{T}_1(u)$ to each other and can thus be considered as generalized commutation relations defining the operator T based on a solution R of the Yang–Baxter equation.

Example 1. Let us consider the most studied example of this realization, namely the case of $\mathfrak{g} = \mathfrak{gl}(N)$ with fundamental generators $\rho(\mathrm{J}^\alpha{}_\beta) = E^\alpha{}_\beta$ and Yang's R-matrix:

$$\mathrm{R}(u-v) = \mathbb{1} + \frac{\mathbb{P}}{u-v}, \tag{3.100}$$

where $\mathbb{P} = \sum_{\alpha,\beta=1}^N E^\alpha{}_\beta \otimes E_\alpha{}^\beta$ again denotes the permutation operator alias the quadratic tensor Casimir operator. We closely follow the lines of [62] which contains an extensive discussion of the RTT-realization for $\mathfrak{gl}(N)$. We expand $\mathrm{T}(u) = \sum_{\alpha\beta} t^\alpha{}_\beta(u) E_\alpha{}^\beta$ as well as (3.94) using that

$$(\mathrm{T}(u) \otimes \mathbb{1})(\mathbb{1} \otimes \mathrm{T}(v)) = \sum_{\alpha,\beta,\gamma,\delta} t^\alpha{}_\beta(u) t^\gamma{}_\delta(v) E_\alpha{}^\beta \otimes E_\gamma{}^\delta, \tag{3.101}$$

$$(\mathbb{1} \otimes \mathrm{T}(v))(\mathrm{T}(u) \otimes \mathbb{1}) = \sum_{\alpha,\beta,\gamma,\delta} t^\gamma{}_\delta(v) t^\alpha{}_\beta(u) E_\alpha{}^\beta \otimes E_\gamma{}^\delta. \tag{3.102}$$

Applying both sides of the RTT-relations to a basis vector $e_\beta \otimes e_\delta \in \mathbb{C}^N \otimes \mathbb{C}^N$ one finds on the left hand side

$$\sum_{\alpha,\gamma} t^\alpha{}_\beta(u) t^\gamma{}_\delta(v) e_\alpha \otimes e_\gamma - \frac{1}{u-v} \sum_{\alpha,\gamma} t^\alpha{}_\beta(u) t^\gamma{}_\delta(v) e_\gamma \otimes e_\alpha, \tag{3.103}$$

and on the right hand side

$$\sum_{\alpha,\gamma} t^\gamma{}_\delta(v) t^\alpha{}_\beta(u) e_\alpha \otimes e_\gamma - \frac{1}{u-v} \sum_{\alpha,\gamma} t^\gamma{}_\beta(v) t^\alpha{}_\delta(u) e_\alpha \otimes e_\gamma. \tag{3.104}$$

Multiplication by $u-v$ and equating the coefficients of independent basis elements $e_\alpha \otimes e_\gamma$ yields

$$(u-v)[t^\alpha{}_\beta(u), t^\gamma{}_\delta(v)] = t^\gamma{}_\beta(u) t^\alpha{}_\delta(v) - t^\gamma{}_\beta(v) t^\alpha{}_\delta(u). \tag{3.105}$$

Expanding as in (3.96) then gives

$$[(t^{(r+1)})^\alpha{}_\beta, (t^{(s)})^\gamma{}_\delta] - [(t^{(r)})^\alpha{}_\beta, (t^{(s+1)})^\gamma{}_\delta] = (t^{(r)})^\gamma{}_\beta (t^{(s)})^\alpha{}_\delta - (t^{(s)})^\gamma{}_\beta (t^{(r)})^\alpha{}_\delta, \tag{3.106}$$

for $r, s = 0, 1, \ldots$ and $(t^{(0)})^\alpha{}_\beta = \delta^\alpha_\beta$. The relations (3.106) may be taken as an alternative definition of the Yangian algebra spanned by $(t^{(r)})^\alpha{}_\beta$. In particular, one typically has the following relation to the Yangian generators in the first realization:[35]

$$(t^{(1)})^\alpha{}_\beta \simeq \mathrm{J}^a \, (t^a)^\alpha{}_\beta \qquad\qquad (t^{(2)})^\alpha{}_\beta \simeq \widehat{\mathrm{J}}^a \, (t^a)^\alpha{}_\beta + \ldots, \tag{3.107}$$

where the dots stand for lower-level generators or the identity, cf. [63].

Example 2. Notably, in the above example we did not require the auxiliary map $c(u)$ of (3.99). Let us thus also consider another example with $\mathfrak{g} = \mathfrak{sl}(N)$ where this map is required, c.f. [6]. Then $Y[\mathfrak{g}]$ is isomorphic to the algebra $\mathfrak{a}_\rho$ defined by the relations

$$\Big(\mathbb{1} + \frac{\mathbb{P}}{u-v}\Big) \mathrm{T}_1(u) \mathrm{T}_2(v) = \mathrm{T}_2(v) \mathrm{T}_1(u) \Big(\mathbb{1} + \frac{\mathbb{P}}{u-v}\Big), \tag{3.108}$$

and the auxiliary constraint equation

$$c(u) = \det{}_{\mathrm{q}} \, \mathrm{T}(u) = 1. \tag{3.109}$$

[35] See Section 5.1 for an expansion of the monodromy $\mathrm{T}(u)$ in the context of spin chains.

Here $\mathbb{P}$ denotes again the permutation operator and the so-called quantum determinant is defined as [64]

$$\det_{\mathrm{q}} \mathrm{T}(u) = \sum_{\mathrm{Perm}(\alpha_1,\ldots,\alpha_n)} \mathrm{sign}(\alpha_1,\ldots,\alpha_n) t^1{}_{\alpha_1}\Big(u+\frac{n-1}{2}\Big) t^2{}_{\alpha_2}\Big(u+\frac{n-3}{2}\Big)\ldots t^n{}_{\alpha_n}\Big(u+\frac{1-n}{2}\Big), \quad (3.110)$$

where the sum runs over permutations of $(\alpha_1,\ldots,\alpha_n)$ with values in $1,\ldots,n$. Note that here $\mathrm{R}(u)$ takes again the form of Yang's R-matrix. In the above sense, the Yangian algebra for $\mathfrak{sl}(N)$ (Example 2) is given by the Yangian of $\mathfrak{gl}(N)$ (Example 1) modulo the quantum determinant relation (3.109).

4 Quantum Nonlocal Charges and Yangian Symmetry in 2d Field Theory

In this section we will rediscover some of the concepts learned about Yangian symmetry in the last chapter. In particular, we will discuss Drinfel'd's first realization.

4.1 Quantum Nonlocal Charges alias Yangian Symmetry

Let us start by noting a crucial difference to the case of classical charges considered in Section 2. In a quantum field theory, the product of two operators $\mathcal{O}_1(x)\mathcal{O}_2(y)$ is typically divergent in the limit $x \to y$. That this is a priori a problem in the context of nonlocal symmetries becomes immediately clear when looking at the classical bilocal current:[36]

$$(\widehat{j}_{\mathrm{classical}})^{\mu}_{a}(x) = \epsilon^{\mu\nu} j_{a\nu}(x) - \tfrac{1}{2} f_{abc} \int_{-\infty}^{x} \mathrm{d}y\, j^{\mu}_{b}(x) j^{0}_{c}(y). \quad (4.1)$$

The current contains the product $j^{\mu}_{b}(x) j^{0}_{c}(y)$ and since y is integrated up to x, the problem is apparent. In order to get control over the divergencies, it is useful to employ the so-called *point-splitting* regularization, i.e. to split the point x into two points x and $x-\delta$. Then the short-distance singularities of the product $\mathcal{O}_1(x)\mathcal{O}_2(x-\delta)$ can be extracted as the coefficients in the expansion around $\delta = 0$. Below we will use this point-splitting regularization in order to define a quantum bilocal current, but first we have to understand the singularities of the current product a bit better.

In general, the question for the behavior of the product of currents in the limit $x \to y$ is addressed by the *operator product expansion* (OPE) which takes the form

$$f_{abc} j^{b}_{\mu}(x) j^{c}_{\nu}(0) = \sum_{i} c^{(i)}_{\mu\nu}(x) \mathcal{O}^{(i)}_{a}(0). \quad (4.2)$$

Here the f_{abc} denote again the structure constants of an internal semi-simple Lie algebra symmetry $\mathfrak{g}$ with level-zero charges J_a induced by the local currents which obey

$$[\mathrm{J}_a, \mathrm{J}_b] = \mathcal{N} f_{abc} \mathrm{J}_c. \qquad f_{abc} f_{bcd} = -\mathcal{C}\delta_{ad}. \quad (4.3)$$

We have introduced a (possibly coupling-dependent) normalization $\mathcal{N}$ and the adjoint Casimir $\mathcal{C}$. Note that understanding the OPE also furnishes a quantum analogue of the classical flatness condition with a proper normal ordering prescription:[37]

$$\partial_0 j^{a}_{1} - \partial_1 j^{a}_{0} + f_{abc} : j^{b}_{0} j^{c}_{1} := 0. \quad (4.4)$$

[36] We assume that the classical conservation of the current is not broken by quantum anomalies. The breaking of symmetries at the quantum level may occur if the symmetry is a symmetry of the action but not of the measure of the path integral.

[37] Cf. the appendix of [65].

Lüscher's theorem. In general, the OPE can be studied by exploiting the fact that both sides of equation (4.2) have to obey the same symmetries and carry the same quantum numbers. With the aim to quantize the definition of the bilocal current $\widehat{j}_\mu$, we follow [36, 11, 65] and try to understand what can be said about the operator product (4.2) if one makes the following set of assumptions for the two-dimensional quantum field theory under consideration:

- The theory is renormalizable (to have a well-defined OPE) and asymptotically free (which determines the scaling behavior of $c^{(i)}_{\mu\nu}(x)$).
- The theory has a local conserved current j^a_μ.
- There is only one operator of dimension smaller than 2 that transforms under the adjoint representation of $\mathfrak{g}$, namely the conserved current $j^a_\mu(x)$ (and derivatives thereof).
- Both sides of (4.2) obey C, P, T and Lorentz symmetry.[38]
- The current commutes with itself when evaluated at different points with spacelike separation (locality).

Under these assumptions it was shown that the most general form of the above OPE is given by [36, 11, 65]

$$f_{abc} j^b_\mu(x) j^c_\nu(0) = c^\rho_{\mu\nu}(x) j^a_\rho(0) + d^{\sigma\rho}_{\mu\nu}(x) \partial_\sigma j^a_\rho(0), \tag{4.5}$$

with the below specifications on the OPE coefficient functions. In order to see that this behavior is compatible with the charge algebra (4.3), one considers the equal time commutator:

$$f_{abc}[j^b_\mu(x), j^c_\nu(0)]_{\text{e.t.}} = \lim_{\epsilon\to 0} f_{abc}\Big[j^b_\mu(x,-i\epsilon) j^c_\nu(0) - j^b_\mu(x,i\epsilon) j^c_\nu(0)\Big]. \tag{4.6}$$

Evaluating the OPE at $-x^2-\epsilon^2$ and with the normalization $\mathcal{N}$ entering by [65]

$$f_{abc}[j^b_0(x), j^c_\mu(0)]_{\text{e.t.}} = -\mathcal{N}\mathcal{C}\delta(x) j_{a,\mu}(0), \tag{4.7}$$

one finds an expansion that sometimes goes under the name *Lüscher's theorem* [36, 11, 65]:[39]

$$c^\rho_{\mu\nu}(x) = a_1(x)\frac{\eta_{\mu\nu}x^\rho}{x^2} + a_2(x)\frac{x_{(\mu}\delta^\rho_{\nu)}}{x^2} + a_3(x)\frac{x_\mu x_\nu x^\rho}{x^4}, \tag{4.8}$$

$$d^{\sigma\rho}_{\mu\nu}(x) = \frac{b_1(x)}{4}\frac{x_{[\mu}x^\rho\delta^\sigma_{\nu]} + x_{[\mu}x^\sigma\delta^\rho_{\nu]}}{x^2} + \frac{b_2(x)}{4}\delta^\sigma_{[\mu}\delta^\rho_{\nu]} + \frac{x^\sigma}{2}c^\rho_{\mu\nu}(x). \tag{4.9}$$

Here all coefficient functions a_k, b_k depend on x only via the Lorentz-invariant x^2 and are of order $\mathcal{O}(|x|^{-0})$ due to the asymptotic freedom of the theory.[40] Furthermore the parameter functions depend on the normalization $\mathcal{N}$, the Casimir $\mathcal{C}$ and on one model-dependent function $\xi(x)$ which is a function of $\log(\mu^2x^2)$, where μ denotes a mass scale. Using the above assumptions one can derive many constraints on the parameter functions. Current conservation translated into $\partial^\mu c^\rho_{\mu\nu} = 0$ and $\partial^\mu d^{\rho\sigma}_{\mu\nu} = 0$ for example implies several differential equations [36, 65, 66], e.g. at $t = 0$:

$$a_1(x) = -2x\frac{\mathrm{d}}{\mathrm{d}x}\Big[b_1(x) - b_2(x)\Big]. \tag{4.10}$$

[38] Here we assume that the theory has a C operation. See [65] for a discussion of its properties in this context.

[39] Lüscher derived this form of the current product for the case of the non-linear sigma model [36], while Bernard obtained similar constraints for the massive current algebras in two dimensions [11]. A nice general discussion is given in [65].

[40] The notation $\mathcal{O}(|x|^{-0})$ denotes possible logarithmic terms.

Evaluating all these constraints yields the relations [36, 65]

$$a_1(x) = -2\,\dot{b}_1(x) - b_1(x) + \frac{\mathcal{NC}}{2\pi}, \quad a_2(x) = b_1(x) - \frac{\mathcal{NC}}{2\pi}, \quad a_3(x) = 2\dot{b}_1(x) - 2b_1(x), \tag{4.11}$$

as well as

$$b_1 = -\dot{\xi}(x) + \frac{\mathcal{NC}}{2\pi}, \qquad\qquad b_2 = \dot{\xi}(x) + \xi(x) - \frac{\mathcal{NC}}{2\pi}, \tag{4.12}$$

where the dot denotes the derivative with respect to $\log(\mu^2 x^2)$.

Quantum bilocal current. We may now introduce the point-split version of the nonlocal current as

$$\widehat{j}^{\mu}_a(t,x|\delta) = Z(\delta)\epsilon^{\mu\nu} j_{a\nu}(t,x) - \tfrac{1}{2} f_{abc} j^{\mu}_b(x) \int_{-\infty}^{x-\delta} \mathrm{d}y\, j^0_c(y), \tag{4.13}$$

where $Z(\delta)$ denotes a renormalization constant that has to be determined. The *quantized bilocal current* is defined as the limit [36, 11]

$$\widehat{j}^{\mu}_a(t,x) = \lim_{\delta\to 0} \widehat{j}^{\mu}_a(t,x|\delta). \tag{4.14}$$

This current is finite and conserved only for a particular choice of the renormalization constant $Z(\delta)$. To understand the divergence, we evaluate the relevant contributions to the product $j_0 j_0$ using (4.5):

$$\begin{aligned} c^{\rho}_{00}(x) &= a_1(x)\frac{x^\rho}{x^2} + 2a_2(x)\frac{t\delta^\rho_0}{x^2} + a_3(x)\frac{t^2 x^\rho}{x^4} \xrightarrow{t\to 0} a_1(x)\delta^\rho_1\frac{1}{x}, \\ d^{\sigma\rho}_{00}(x) &= \tfrac{1}{2}x^\sigma c^\rho_{00}(x) \xrightarrow{t\to 0} a_1(x)\delta^\rho_1\delta^\sigma_1. \end{aligned} \tag{4.15}$$

This indicates that the origin of the divergence for $x \to 0$ lies in the term proportional to $a_1(x)$ in the first line. The bilocal level-one charge is given by

$$\widehat{\mathrm{J}}(\delta) = \int_{-\infty}^{\infty} \mathrm{d}x\, \widehat{j}^0(t,x|\delta), \tag{4.16}$$

and the divergent part has the form (cf. [66])

$$\widehat{\mathrm{J}}(\delta) \simeq Z(\delta) \int_{-\infty}^{\infty} \mathrm{d}y\, j_1(y) - \tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}x \int_{-\infty}^{x-\delta} \mathrm{d}y\, \frac{a_1(x-y)}{x-y} j_1(x-y). \tag{4.17}$$

Now we may use (4.10) to rewrite this as

$$\begin{aligned} \widehat{\mathrm{J}}(\delta) &\simeq Z(\delta)\int_{-\infty}^{\infty}\mathrm{d}y\, j_1(y) + \int_{-\infty}^{\infty}\mathrm{d}x\, j_1(x)\int_{-\infty}^{x-\delta}\mathrm{d}y\,\frac{\mathrm{d}}{\mathrm{d}y}\Big[b_1(x-y) - b_2(x-y)\Big] \\ &\simeq Z(\delta)\int_{-\infty}^{\infty}\mathrm{d}y\, j_1(y) + \int_{-\infty}^{\infty}\mathrm{d}x\, j_1(x)\Big[b_1(x-y) - b_2(x-y)\Big]^{y=x-\delta}_{y=-\infty} \\ &\simeq Z(\delta)\int_{-\infty}^{\infty}\mathrm{d}y\, j_1(y) + \Big[b_1(\delta) - b_2(\delta)\Big]\int_{-\infty}^{\infty}\mathrm{d}x\, j_1(x). \end{aligned} \tag{4.18}$$

Note that $b_1(\delta)$ and $b_2(\delta)$ are divergent for $\delta \to 0$ and that the terms proportional to $+b_1(x+\infty)-b_2(x+\infty)$ should be finite by the conditions on the conserved local current at the boundaries of space. The renormalization constant $Z(\delta)$ is determined by requiring that the bilocal current is finite in the limit $\delta \to 0$ and thus

$$Z(\delta) \equiv b_2(\delta) - b_1(\delta) = 2\dot{\xi}(\delta) + \xi(\delta) - \frac{\mathcal{N}\mathcal{C}}{\pi}. \tag{4.19}$$

Similarly one may show that the quantum bilocal charge induced by the current (4.14) is *conserved* under the above assumptions [36, 65].

Quantum monodromy and Lax formulation. As seen in the classical case, the existence of nonlocal charges in principle allows to define a conserved generating function. For a large class of models, in [42] the quantum analogue of the monodromy matrix $\mathrm{T}(u)$ was constructed directly on asymptotic particle states under the following assumptions:

- A quantum operator $\mathrm{T}(u)$ exists and is conserved.
- $\mathrm{T}(u)$ satisfies a quantum factorization principle (the RTT relations).
- The discrete parity and time-reversal symmetries are realized in the quantum theory.

Since the monodromy matrix $\mathrm{T}(u)$ provides a generating function for the nonlocal conserved charges, it furnishes an alternative way to study the symmetry constraints on observables such as the scattering matrix. However, we will not discuss this in more detail here.

Chiral Gross–Neveu Model. The chiral Gross–Neveu model represents a renormalizable and asymptotically free theory with the symmetry properties assumed above. We may thus apply the quantization procedure to the bilocal current. In order to explicitly compute the OPE expansion, one may insert the current product $f_{abc}j^b_\mu j^c_\nu$ into correlation functions that can be perturbatively evaluated and regularized by ordinary field theory methods. The model-dependent function $\xi(x)$ is given by [65]

$$\xi(x) = \tfrac{\mathcal{N}\mathcal{C}}{2\pi}\log(\mu^2 x^2), \tag{4.20}$$

and thus the renormalization constant evaluates to

$$Z(\delta) = \frac{\mathcal{N}\mathcal{C}}{2\pi}\log(\mu^2\delta^2) + \mathcal{O}(|\delta|^{1-0}) \simeq g^2\frac{N}{2\pi}\log(\mu^2\delta^2) + \mathcal{O}(|\delta|^{1-0}). \tag{4.21}$$

Here we assume the normalization to scale as $\mathcal{N} \simeq g^2$ and the generators of $\mathfrak{su}(N)$ to be normalized such that the adjoint Casimir goes as $\mathcal{C} \simeq N$. The quantum currents for the chiral Gross–Neveu model thus take the form given in [42]:

$$\mathrm{J}_a = \int_{-\infty}^{\infty} \mathrm{d}x\, j^0_a(x), \qquad \widehat{\mathrm{J}}_a = \lim_{\delta\to 0}\Big[Z(\delta)\int_{-\infty}^{\infty}\mathrm{d}x\, j^1_a(x) - \tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{x-\delta}\mathrm{d}x\,\mathrm{d}y\,[j^0(x), j^0(y)]_a\Big]. \tag{4.22}$$

Above the generated mass scale $\mu = \frac{1}{2}me^{\gamma_{\mathrm{E}}}$ is related to the mass m of the fundamental fermions of the chiral Gross–Neveu model and γ_{E} denotes the Euler constant.

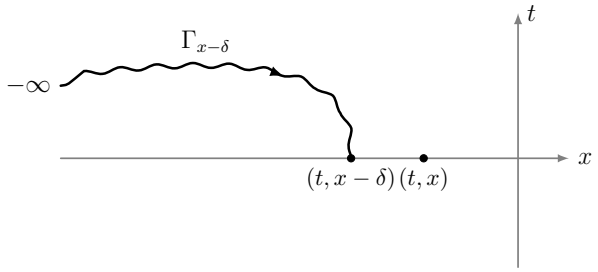


Figure 4: Contour $\Gamma_{x-\delta}$ defining the level-one current.

4.2 Boost Automorphism

Let us understand in some more detail how the abstract mathematical concepts underlying the Yangian algebra are realized in physical theories. We will see that this Hopf algebra in fact circumvents some naive expectations on the symmetry structure of physical observables:

> *"We prove a new theorem on the impossibility of combining spacetime and internal symmetries in any but a trivial way."* S. Coleman and J. Mandula 1967 [10][41]

Under a finite Lorentz transformation, the local current transforms according to

$$U(\Lambda) j^{\mu}(x) U(\Lambda)^{-1} = \Lambda^{\mu}{}_{\nu}\, j^{\nu}(\Lambda x), \tag{4.23}$$

with

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \cosh(u) & \sinh(u) \\ \sinh(u) & \cosh(u) \end{pmatrix}. \tag{4.24}$$

From this one can read off the following vector transformation rule under the boost generator $\mathcal{B} = \frac{\mathrm{d}}{\mathrm{d}u} U(\Lambda)|_{u=0}$:

$$[\mathcal{B}, j_a^{\mu}(x)] = \epsilon_{\rho\sigma} x^{\rho} \partial^{\sigma} j_a^{\mu}(x) + \epsilon^{\mu\sigma} j_{a,\sigma}(x). \tag{4.25}$$

For the transformation of the local charge this straightforwardly implies

$$[\mathcal{B}, \mathrm{J}_a] = 0, \tag{4.26}$$

if $j_a(\pm\infty) = 0$.

Level-one charges and contours. In order to understand the transformation behavior of the bilocal level-one current we use a nice geometric argument of [11]. Note that instead of integrating over the real axis, we may define the bilocal current via integration over a generic contour $\Gamma_{x-\delta}$ that starts at $-\infty$ and ends at $x-\delta$, see Figure 4. This is possible since the current defines a flat connection and thus the integration is path-independent. The bilocal current then takes the form

$$\widehat{j}_a^{\mu}(x, t|\delta) = Z(\delta)\epsilon^{\mu\nu} j_{a\nu}(x,t) - \tfrac{1}{2} f_{abc} j_b^{\mu}(x) \int\limits_{\Gamma_{x-\delta}} \epsilon_{\sigma\rho} \mathrm{d}y^{\rho}\, j_c^{\sigma}(y). \tag{4.27}$$

Here $\epsilon_{\sigma\rho}\mathrm{d}y^{\rho}\, j_c^{\sigma}$ represents the generalization of $\mathrm{d}y j_c^0$ when going away from $t = 0$.

[41] Apparently the assumptions of their famous theorem are not satisfied here. Nevertheless the Coleman–Mandula theorem shows that it is by no means obvious that spacetime and internal symmetries can be nontrivially related.

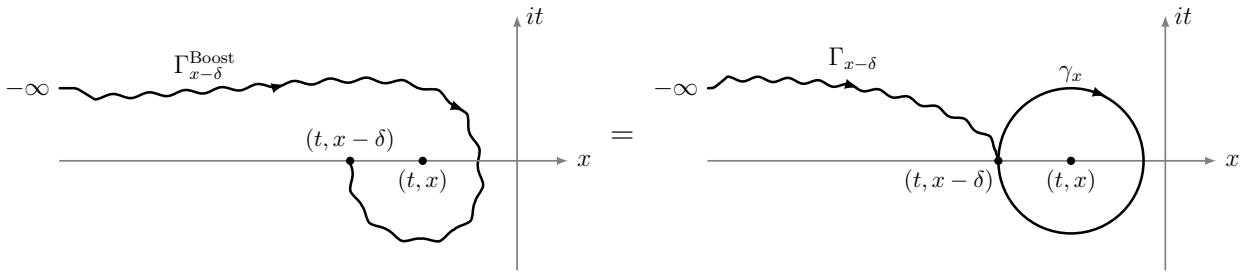


Figure 5: Boost transformation of the contour $\Gamma_{x-\delta}$ by a rapidity $u = 2\pi i$ corresponding to a Euclidean rotation.

One may now apply a Lorentz boost $\mathcal{B}_{2\pi i}$ by an imaginary rapidity $2\pi i$ to this nonlocal current. This boost corresponds to a Euclidean rotation by an angle 2π. As illustrated in Figure 5, the rotated current can be written in the original form with a contour $\Gamma_{x-\delta}$ ending at the point $x - \delta$ plus an integral around the point x over a closed contour γ_x:

$$e^{2\pi i\mathcal{B}}\,\widehat{j}^{\,\mu}_a(x,t|\delta)\,e^{-2\pi i\mathcal{B}} = \widehat{j}^{\,\mu}_a(x,t|\delta) + \tfrac{1}{2}f_{abc}\oint_{\gamma_x}\epsilon_{\sigma\rho}\mathrm{d}y^\rho\, j^\mu_b(x)j^\sigma_c(y). \tag{4.28}$$

We are interested in the implications of this transformation behaviour on the level-one *charges*. Hence, we have to integrate the zero-component of the above expression over x. Then the last integral picks up the residue of the OPE of the product of currents and we would have to make an analysis similar to the one in Section 4.1, where we evaluated the generic structure of the current OPE. We will not discuss this proof in more detail here but note that Bernard has shown that [11]

$$e^{2\pi i\mathcal{B}}\,\widehat{\mathrm{J}}_a e^{-2\pi i\mathcal{B}} = \widehat{\mathrm{J}}_a - \tfrac{1}{2}\mathcal{C}\mathrm{J}_a, \tag{4.29}$$

with $\mathcal{C}$ representing the quadratic Casimir of $\mathfrak{g}$ in the adjoint representation, i.e. $-\delta_{ab}\mathcal{C} = f_{acd}f_{cdb}$. Comparing with the leading orders of the expansion

$$\mathcal{B}_{2\pi i} = \exp(2\pi i\mathcal{B}) = 1 + 2\pi i\mathcal{B} + \ldots, \tag{4.30}$$

one concludes that the conserved charges transform under the boost generator as[42]

$$[\mathcal{B}, \mathrm{J}_a] = 0, \qquad\qquad [\mathcal{B}, \widehat{\mathrm{J}}_a] = -\frac{\mathcal{C}}{4\pi i}\mathrm{J}_a. \tag{4.32}$$

Notably, the nontriviality of the second commutator is a quantum effect (cf. (2.53)). It is induced by the pole in the current OPE encircled by the contour in Figure 5. The operator $\mathcal{B}_u = \exp(u\mathcal{B})$ corresponds to a group-like finite boost transformation whereas the generator $\mathcal{B}$ represents the algebra element whose primitive coproduct follows from the expansion

$$\mathcal{B}_u \otimes \mathcal{B}_u = \mathbb{1}\otimes\mathbb{1} + u(\mathcal{B}\otimes\mathbb{1} + \mathbb{1}\otimes\mathcal{B}) + \mathcal{O}(u^2). \tag{4.33}$$

Hence, the boost transformation couples the internal and spacetime symmetries to each other, which implies that the quantized Yangian algebra is not merely an internal symmetry. It is

[42] In an alternative approach using form factors, the commutation relations with the boost operator were obtained by first determining the commutator of the level-one charge with the energy momentum tensor for a model with $\mathfrak{g} = \mathfrak{sl}(2)$ [12]:

$$[\widehat{\mathrm{J}}^a, T_{\mu\nu}(x)] = -\frac{\mathcal{C}}{8\pi i}\big(\epsilon_{\mu\alpha}\partial_\alpha j^a_\nu(x) + \epsilon_{\nu\alpha}\partial_\alpha j^a_\mu(x)\big). \tag{4.31}$$

Integrating the 00-component one finds the relation (4.32) with the boost generator $\mathcal{B} = \int \mathrm{d}x\, x\, T_{00}$ for $t = 0$.

conceivable that this yields stronger constraints on symmetry invariants than a direct product of independent symmetry algebras. Note that the above boost tranformation on the conserved charges exactly realizes the boost automorphism defined by Drinfel'd. Hence, the internal level-zero and level-one charges J_a and $\widehat{\mathrm{J}}_a$ together with the boost automorphism $\mathcal{B}_u$ furnish the defining relations of the Yangian algebra in the first realization as given in Section 3.1.

> *"Thus one concludes that the Yangian must actually be extended to include the Poincaré algebra with generators $\mathcal{B}, \mathrm{P}_\mu$ in order to realize its full implications."* A. LeClair and F.A. Smirnov 1991 [12]

4.3 Yangian Symmetry and the 2d S-matrix

In physics, symmetries are typically used as a guiding principle to construct new models and to constrain their observables. Being an infinite-dimensional symmetry algebra, the Yangian has strong implications for the spectrum and dynamics of a theory. In [67], Belavin showed for instance that the spectrum of masses of a two-dimensional quantum field theory can be computed via the Yangian symmetry. The prime example for the physical application of the Yangian is the scattering matrix of massive, relativistic, two-dimensional quantum field theories. The S-matrix is the operator that relates asymptotic particles to each other. Since particles are defined by representations of symmetries (e.g. Poincaré symmetry), the scattering matrix is constrained by these symmetries. In a theory with Yangian symmetry, particles also transform in representations of the Yangian, which should thus be a symmetry of the S-matrix, cf. e.g. [12,68].

In order to study scattering processes in two dimensions, it is useful to consider lightcone coordinates defined by

$$p^+ = p^0 + p^1, \qquad p^- = p^0 - p^1, \tag{4.34}$$

which can be expressed in terms of the often more convenient rapidities u by the relations

$$p^+ = m\exp(+u), \qquad p^- = m\exp(-u). \tag{4.35}$$

Here m denotes the particle mass. The two-dimensional momentum in the original coordinates then takes the form

$$(p^\mu) = \begin{pmatrix} m\cosh u \\ m\sinh u \end{pmatrix}, \tag{4.36}$$

which shows that the transformation $u \to -u$ inverts the direction of the particle's movement. The transformation $u \to i\pi - u$ flips the sign of the particle's energy and thus represents a particle to antiparticle transformation. Importantly, the Lorentz boost acts additively on the variables u (see below), which emphasizes their usefulness in the present context.

Poincaré symmetry and scattering states. We will now be interested in understanding the impact of symmetries on the scattering in this theory. We consider the Poincaré algebra in 1+1 dimensions

$$[\mathrm{P}^+, \mathrm{P}^-] = 0, \qquad [\mathcal{B}, \mathrm{P}^+] = +\mathrm{P}^+, \qquad [\mathcal{B}, \mathrm{P}^-] = -\mathrm{P}^-, \tag{4.37}$$

where the single Lorentz transformation in two dimensions is generated by the boost $\mathcal{B}$ and $\mathrm{P}^\pm$ denotes translations into the lightcone directions. In order to study the scattering of different particle species moving in one space dimension, we look at asymptotic scattering states. For

one single particle of type α this state is denoted by $|\alpha, u\rangle$. In the chiral Gross–Neveu model for instance, it takes the form (with $p_1 = p_1(u)$) [42]

$$|\alpha, u\rangle = (p_1^2 + m^2)^{\frac{1}{4}}\, b_\alpha^\dagger(p_1)\,|0\rangle\,, \tag{4.38}$$

where b_α, $b_\alpha^\dagger$ are oscillators that appear in the Fourier decomposition of the fundamental fermions.

Note that the Lorentz boost can be realized on one-particle states (on-shell) as

$$\mathcal{B}\,|\alpha, u\rangle = \frac{\partial}{\partial u}\,|\alpha, u\rangle\,, \tag{4.39}$$

and a finite boost transformation $\mathcal{B}_v = \exp(v\mathcal{B})$ acts on a one-particle state by a shift of the rapidity:

$$\mathcal{B}_v\,|\alpha, u\rangle = |\alpha, u + v\rangle\,. \tag{4.40}$$

The energy-momentum generators act on one-particle states as

$$\mathrm{P}^+\,|\alpha, u\rangle = m\,e^{+u}\,|\alpha, u\rangle\,, \qquad\qquad \mathrm{P}^-\,|\alpha, u\rangle = m\,e^{-u}\,|\alpha, u\rangle\,. \tag{4.41}$$

In order to study the scattering of multiple particles, we need a notion of multi-particle scattering states. Importantly, the fact that the space is one-dimensional allows to order particles (i.e. wave-packets) with respect to their position. It is thus natural to label the particles $1, \dots, n$ according to their space coordinate $x_1 < \cdots < x_n$. However, only if the fastest particle of the incoming multi-particle state is on the very left, it can cross all other particle trajectories. Thus, in order to have a nontrivial n-particle scattering process, the particle rapidities u_k in the in-state have to have the opposite ordering as compared to the positions, i.e. $u_1 > \cdots > u_n$. After the scattering process, the situation is reversed, and the particles in the out-state with positions $x_1' < \cdots < x_n'$ have rapidities ordered as $u_1' < \cdots < u_n'$. This motivates to introduce the following ordered multi-particle states with rapidities ordered as $u_1 > \cdots > u_n$:[43]

$$|\alpha_1, u_1; \dots; \alpha_n, u_n\rangle_{\text{in}}\,, \qquad\qquad x_1 < x_2 < \cdots < x_n, \tag{4.42}$$

$$|\beta_1, u_1; \dots; \beta_n, u_n\rangle_{\text{out}}\,. \qquad\qquad x_1 > x_2 > \cdots > x_n. \tag{4.43}$$

In fact, the scattering matrix in 1+1 dimensions is defined as the operator that expresses an out-state in the infinite future, in the in-state basis in the infinite past, or vice versa. We assume that the scattering process preserves the number of particles as well as the individual rapidities, as is the case in integrable theories in two dimensions.[44] Then the S-matrix acts on the asymptotic states as

$$|\alpha_1, u_1; \dots; \alpha_n, u_n\rangle_{\text{in}} = \mathrm{S}_{\alpha_1,\dots,\alpha_n}^{\beta_1,\dots,\beta_n}(u_1, \dots, u_n)\,|\beta_1, u_1; \dots; \beta_n, u_n\rangle_{\text{out}}\,. \tag{4.44}$$

Note that the above definition of in- and out-states implies that the coproduct acts differently on the two bases. This serves as a motivation to introduce the notion of *opposite coproduct* for the permuted coproduct (3.18), which enters the below symmetry equation for the S-matrix. The action of the above Poincaré generators on one-particle states generalizes to multi-particle states via the primitive coproduct

$$\Delta(\mathrm{P}^+) = \mathrm{P}^+ \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{P}^+, \quad \Delta(\mathrm{P}^-) = \mathrm{P}^- \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{P}^-, \quad \Delta(\mathcal{B}) = \mathcal{B} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{B}. \tag{4.45}$$

[43] The different ordering of positions corresponds to different orders of the operators generating the individual particles from the vaccuum, cf. (4.38). In fact, the S-matrix theory can be formulated in terms of such operators spanning the so-called Zamolodchikov–Faddeev algebra.

[44] Also this can be shown using the Yangian structure of the S-matrix but we do not discuss this here.

Yangian symmetry and scattering states. Suppose now that in addition to the above Poincaré symmetry, the underlying theory features a Yangian extension $Y[\mathfrak{g}]$ of an internal symmetry algebra $\mathfrak{g}$. Most of the commutators between the spacetime and internal symmetries vanish, but as seen above, the boost operator has nontrivial commutation relations with the Yangian level-one generators:

$$[\mathrm{P}^{\pm}, \mathrm{J}_a] = 0, \qquad [\mathrm{P}^{\pm}, \widehat{\mathrm{J}}_a] = 0, \qquad [\mathcal{B}, \mathrm{J}_a] = 0, \qquad [\mathcal{B}, \widehat{\mathrm{J}}_a] = -\mathcal{C}\frac{\hbar}{4\pi i}\mathrm{J}_a, . \tag{4.46}$$

Thus the Lorentz boost furnishes the boost automorphism of the Yangian algebra. The above relations imply that the one-particle states transform in an evaluation representation of the Yangian:

$$\rho_{\hat{u}}(\mathrm{J}_a)\,|u\rangle = \rho_0(\mathrm{J}_a)\,|u\rangle\,, \qquad\qquad \rho_{\hat{u}}(\widehat{\mathrm{J}}_a)\,|u\rangle = \hat{u}\,\rho_0(\mathrm{J}_a)\,|u\rangle\,. \tag{4.47}$$

Here ρ_0 again denotes a representation of the generators of our symmetry algebra acting on the particle labels $\alpha, \beta, \ldots$ (cf. (3.49)) and $\hat{u}$ equals the rapidity u up to a constant:

$$\hat{u} = -\frac{\hbar\mathcal{C}}{4\pi i}u. \tag{4.48}$$

The compatibility of this representation with the boost commutator in (4.46) can be seen by evaluating

$$\rho([\mathcal{B}, \widehat{\mathrm{J}}_a])\,|u\rangle = \left[\partial_u\rho_0(\widehat{\mathrm{J}}_a) - \rho_0(\widehat{\mathrm{J}}_a)\partial_u\right]|u\rangle = -\frac{\hbar\mathcal{C}}{4\pi i}\rho_0(\mathrm{J}_a)\,|u\rangle\,. \tag{4.49}$$

or alternatively (cf. (3.24))[45]

$$\mathcal{B}_v(\widehat{\mathrm{J}}_a)\,|u\rangle \equiv e^{v\mathcal{B}}\widehat{\mathrm{J}}_a e^{-v\mathcal{B}}\,|u\rangle = \left(\widehat{\mathrm{J}}_a - \frac{\hbar\mathcal{C}}{4\pi i}\mathrm{J}_a v\right)|u\rangle\,, \tag{4.50}$$

where the expansion of $\exp(v\mathcal{B})$ yields only finitely many terms since $[\mathcal{B}, \mathrm{J}_a] = 0$. In order to study the implications of the symmetry on the scattering matrix, we note that the conserved charges are time-independent and are thus the same on incoming and outgoing states, e.g. for the level-one charge [36]:

$$\widehat{\mathrm{J}}^a = \lim_{t\to-\infty}\widehat{\mathrm{J}}^a_{\mathrm{in}}(t) = \lim_{t\to\infty}\widehat{\mathrm{J}}^a_{\mathrm{out}}(t). \tag{4.51}$$

The action on multiparticle states is defined by the coproduct. Each particle transforms with a different evaluation parameter $\hat{u}_i$ such that we find

$$\begin{aligned}
\rho_{\hat{u}}(\mathrm{J}_a)\,|u_1,\ldots,u_n\rangle_{\substack{\mathrm{in}\\\mathrm{out}}} &= \sum_{k=1}^{n}\rho_0(\mathrm{J}_{a,k})\,|u_1,\ldots,u_n\rangle_{\substack{\mathrm{in}\\\mathrm{out}}}\,, \\
\rho_{\hat{u}}(\widehat{\mathrm{J}}_a)\,|u_1,\ldots,u_n\rangle_{\substack{\mathrm{in}\\\mathrm{out}}} &= \Big(\sum_{k=1}^{n}\hat{u}_k\,\rho_0(\mathrm{J}_{a,k}) \pm \tfrac{1}{2}f_{abc}\sum_{1\le i<j\le n}\rho_0(\mathrm{J}_{b,i})\rho_0(\mathrm{J}_{c,j})\Big)\,|u_1,\ldots,u_n\rangle_{\substack{\mathrm{in}\\\mathrm{out}}}\,,
\end{aligned} \tag{4.52}$$

where we have dropped the particle flavors for simplicity of the expression. The different signs $\pm$ arise from the application of the different coproducts Δ or Δ^{op} to the out- or in-state, respectively.

[45] Note that $\mathcal{B}_v$ acts on an operator by conjugation with $\exp(v\mathcal{B})$.

Constraints on the S-matrix. We would like to understand the two-particle S-matrix, which typically serves as the fundamental building block for integrable scattering matrices in two dimensions:

$$|\alpha_1, u_1; \alpha_2, u_2\rangle_{\text{in}} = \mathrm{S}_{\alpha_1,\alpha_2}^{\beta_1,\beta_2}(u_1, u_2)\,|\beta_1, u_1; \beta_2, u_2\rangle_{\text{out}}\,. \tag{4.53}$$

Lorentz symmetry is the statement $[\mathcal{B}\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{B}, \mathrm{S}(u_1,u_2)]=0$, or explicitly

$$\frac{\partial}{\partial u_1}\mathrm{S}(u_1,u_2)+\frac{\partial}{\partial u_2}\mathrm{S}(u_1,u_2)=0, \tag{4.54}$$

which is solved by $\mathrm{S}(u_1,u_2)=\mathrm{S}(u_1-u_2)$. This is equivalent to saying that the two-particle S-matrix may only depend on the Mandelstam variable $(p_1+p_2)^2=4m^2\cosh^2(\frac{u_1-u_2}{2})$.

Let us emphasize that the scattering matrix is the operator that relates the in- and out-representation in a scattering process:

$$|\ldots\rangle_{\text{in}} = \mathrm{S}\,|\ldots\rangle_{\text{out}}\,. \tag{4.55}$$

The out- and in-representations transform under different coproducts, namely under Δ and Δ^{op} (cf. (3.18) and the definition of multi-particle states (4.42)) and hence, the internal symmetry of the theory implies that the S-matrix furnishes an intertwiner for the Yangian evaluation modules:[46]

$$\rho\Big(\mathcal{B}_{\hat{u}}\otimes\mathcal{B}_{\hat{v}}(\Delta^{\text{op}}(a))\Big)\mathrm{S}(u-v)=\mathrm{S}(u-v)\rho\Big(\mathcal{B}_{\hat{u}}\otimes\mathcal{B}_{\hat{v}}(\Delta(a))\Big), \tag{4.57}$$

for all $a\in Y[\mathfrak{g}]$. In order to remember the explicit relation of the boost to the evaluation representation, we consider again the two-particle expressions for the level-zero and level-one generators:

$$\mathcal{B}_u\otimes\mathcal{B}_v(\Delta(\mathrm{J}_a)) = \mathrm{J}_a\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{J}_a. \tag{4.58}$$

$$\mathcal{B}_u\otimes\mathcal{B}_v(\Delta(\widehat{\mathrm{J}}_a)) = \widehat{\mathrm{J}}_a\otimes\mathbb{1}+\mathbb{1}\otimes\widehat{\mathrm{J}}_a+u(\mathrm{J}_a\otimes\mathbb{1})+v(\mathbb{1}\otimes\mathrm{J}_a)-\tfrac{1}{2}f_{abc}\mathrm{J}_b\otimes\mathrm{J}_c. \tag{4.59}$$

Notably, (4.57) represents (3.27) realized on the scattering matrix. For the level-zero and level-one generators we therefore have via (3.32) and (3.33) of Theorem 2, and using the representation (3.49):

$$0=[\rho(\mathrm{J}_a)\otimes\mathbb{1}+\mathbb{1}\otimes\rho(\mathrm{J}_a), \mathrm{S}(u_1-u_2)], \tag{4.60}$$

$$0=(\rho_0\otimes\rho_0)\Big(\mathcal{B}_{\hat{u}}\otimes\mathcal{B}_{\hat{v}}(\Delta^{\text{op}}(\widehat{\mathrm{J}}_a))\Big)\mathrm{S}(u-v)-\mathrm{S}(u-v)(\rho_0\otimes\rho_0)\Big(\mathcal{B}_{\hat{u}}\otimes\mathcal{B}_{\hat{v}}(\Delta(\widehat{\mathrm{J}}_a))\Big). \tag{4.61}$$

We thus conclude, that the above constraints following from Yangian symmetry imply that the two-particle scattering matrix in our 1+1 dimensional field theory satisfies the quantum Yang–Baxter equation

$$\mathrm{S}_{12}(u_{12})\mathrm{S}_{13}(u_{13})\mathrm{S}_{23}(u_{23})=\mathrm{S}_{23}(u_{23})\mathrm{S}_{13}(u_{13})\mathrm{S}_{12}(u_{12}), \tag{4.62}$$

which allows to consistently factorize multi-particle scattering into two-particle S-matrices. This property represents the hallmark of an integrable theory in two dimensions. As an explicit example, we have considered the solution (3.41) to the above constraint equations for $Y[\mathfrak{su}(2)]$. In order to fix the scalar prefactor of the S-matrix, one has to impose further symmetry properties such as crossing and unitarity which also follow from the Yangian Hopf algebra, but which will not be discussed here in further detail. For a more detailed discussion of the S-matrix see for instance [70, 28].

[46] Sometimes the matrix $\check{\mathrm{S}}=\mathbb{P}\,\mathrm{S}$ is called S-matrix, for which this condition becomes (see e.g. [69])

$$\rho\big(\mathcal{B}_{\hat{v}}\otimes\mathcal{B}_{\hat{u}}(\Delta(a))\big)\check{\mathrm{S}}(u-v)=\check{\mathrm{S}}(u-v)\rho\big(\mathcal{B}_{\hat{u}}\otimes\mathcal{B}_{\hat{v}}(\Delta(a))\big). \tag{4.56}$$

5 Spin Chains and Discrete Yangian Symmetry

As opposed to the continuous theories discussed above, Yangian symmetry also plays an important role in 1+1 dimensional models with a discrete space dimension. This chapter is concerned with such models on a one-dimensional lattice which are called *spin chains*. Many of the features of the Yangian have a discrete nature (e.g. the coproduct) or take over to the discrete case by replacing continuous integrals over space by discrete sums. On the other hand there are important differences to the continuous field theories. Depending on your background, spin chains may often be the most accessible framework to discuss Yangian symmetry.

Spin chains and local charges. From quantum mechanics the spin associated to the algebra $\mathfrak{g} = \mathfrak{su}(2)$ is well-known. For the case of spin $\frac{1}{2}$ for instance, the spin can be considered as a vector space that transforms under the fundamental representation of $\mathfrak{su}(2)$. It may take the orientations up $|\uparrow\rangle$ or down $|\downarrow\rangle$.

We consider a (generalized) spin as a vector space $\mathbb{V}$ that transforms under some representation of a symmetry algebra $\mathfrak{g}$. Now we may go further and form chains of spins. We define such *spin chains* as physical models on a Hilbert space $\mathbb{H}$, which is a tensor product of the above vector spaces:

$$\mathbb{H} = \ldots \otimes \mathbb{V}_k \otimes \mathbb{V}_{k+1} \otimes \mathbb{V}_{k+2} \otimes \ldots . \tag{5.1}$$

Here we will assume that all vector spaces are identical $\mathbb{V}_k = \mathbb{V}$. The index k labels the position or site of the spin chain and the positions k and $k+1$ are called *nearest neighbors*. The chain may have different boundary conditions, e.g. periodic, open, infinite or semi-infinite boundary conditions, which we leave unspecified for the moment. We will briefly discuss different boundary conditions and Yangian symmetry in Section 5.2.

The spin chain Hilbert space $\mathbb{H}$ is spanned by states for which the spin at each position k has a fixed orientation v_k, where v_k denotes a basis vector of $\mathbb{V}$:

$$|\ldots, v_k, v_{k+1}, v_{k+2}, \ldots\rangle \in \mathbb{H}. \tag{5.2}$$

A physical model is typically defined by a local Hamiltonian H, whose density acts on two neighboring sites or so-called nearest neighbors. In the case of integrable spin chains, one finds a set of integrable charges or higher Hamiltonians $\mathcal{Q}_n$, with the first charge given by the two-site or nearest-neighbor Hamiltonian $\mathrm{H} = \mathcal{Q}_2$. The integrability of the model is reflected in the fact that all of the charges mutually commute:

$$[\mathcal{Q}_m, \mathcal{Q}_n] = 0, \qquad m, n = 2, 3, \ldots . \tag{5.3}$$

These charge operators usually act *locally* and *homogeneously* on spin chains; this means that their density acts merely on a small number of neighboring sites and the form of the interaction encoded in the density $\mathcal{Q}_{n,k} \equiv \mathcal{Q}_n(k)$ does not depend on the position k, respectively. We will focus on such charge operators being *invariant* under the symmetry $\mathfrak{g}$ and acting on the spin chain as

$$\mathcal{Q}_n := \sum_k \mathcal{Q}_{n,k}, \qquad \mathcal{Q}_{n,k} : \mathbb{V}_k \otimes \ldots \otimes \mathbb{V}_{k+n-1} \to \mathbb{V}_k \otimes \ldots \otimes \mathbb{V}_{k+n-1}. \tag{5.4}$$

Here the density $\mathcal{Q}_{n,k}$ is a linear operator which acts on several consecutive spins starting with site k, cf. Figure 6. The number n of interacting sites is called the interaction range of the operator $\mathcal{Q}_n$. For most ordinary spin chains the charges are labelled such that the interaction range of $\mathcal{Q}_n$ is indeed n.

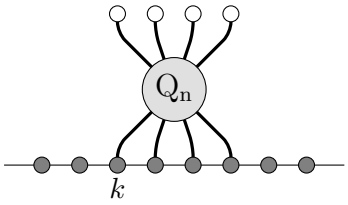
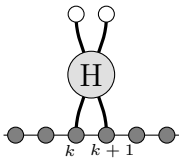


Figure 6: A local charge operator $\mathcal{Q}_n$ acting on a spin chain (here $n = 4$). Its position k on the chain is summed over, see (5.4).

A simple example for an integrable spin chain model is the Heisenberg or $\mathrm{XXX}_{\frac{1}{2}}$ spin chain[47] with $\mathfrak{g} = \mathfrak{su}(2)$ symmetry. Its Hamiltonian is given by the local operator

$$\mathrm{H}_{\mathrm{XXX}} = \sum_k \mathrm{H}_{k,k+1} = \tfrac{1}{2}\sum_k (\mathbb{1}_k \otimes \mathbb{1}_{k+1} - \sigma^a_k \otimes \sigma^a_{k+1}) = \sum_k$$ 

(5.5)

with σ^a_k denoting the Pauli matrices acting on site k of the spin chain. Alternatively, the Hamiltonian can be expressed in terms of the permutation operator using again that the tensor Casimir of $\mathfrak{u}(2)$ takes the form $\mathbb{P}_{k,k+1} = \frac{1}{2}(\mathbb{1}_k \otimes \mathbb{1}_{k+1} + \sigma^a_k \otimes \sigma^a_{k+1})$:

$$\mathrm{H}_{\mathrm{XXX}} = \sum_k (\mathbb{1}_{k,k+1} - \mathbb{P}_{k,k+1}). \tag{5.6}$$

It is convenient to introduce an even more compact notation by writing $\mathrm{H}_{\mathrm{XXX}} = [1,2] - [2,1]$. This square bracket notation straightforwardly generalizes to permutations of higher range:

$$[a_1, a_2, \ldots, a_\ell]\,|X_1, X_2, \ldots, X_L\rangle = \sum_k |X_1, \ldots, X_k, X_{k+a_1}, \ldots, X_{k+a_n}, X_{k+\ell+1}, \ldots, X_L\rangle. \tag{5.7}$$

e.g. for the permutation operator we have

$$\mathbb{P}\,|X_1, \ldots, X_L\rangle \equiv [2,1]|X_1, \ldots, X_L\rangle = \sum_k |X_1, \ldots, X_k, X_{k+2}, X_{k+1}, X_{k+3}, \ldots, X_L\rangle. \tag{5.8}$$

In the above expression (5.7), the limits of the sum over k depend on the boundary conditions. For periodic boundary conditions we have $\sum_{k=1}^{L}$, for open boundary conditions $\sum_{k=1}^{L+1-\ell}$, and for infinite boundary conditions we have $\sum_{k=-\infty}^{\infty}$.

5.1 Lax Operator, Monodromy and Yangian Generators

Consider a spin chain with spins $|X_k\rangle \in \mathbb{V}_k$, where we assume that all *physical* or *quantum spaces* $\mathbb{V}_k = \mathbb{V}$ are identical. The algebraic construction of spin chain models employs the concept of a so-called *auxiliary space* $\mathbb{V}_0$. The auxiliary space typically transforms under the fundamental representation of the symmetry algebra and we label it with an index 0 or $\bar{0}$ in order to distinguish it from the physical spaces.

As a generalization of the continuous classical case (2.43), we introduce a *Lax operator* on the space. This Lax operator or Lax matrix acts on a physical space $\mathbb{V}_k$ and on an auxiliary

[47] The name XXX stems from the fact that the coefficients in the Hamiltonian (5.5) of $\sigma^1_k \otimes \sigma^1_{k+1}$, $\sigma^2_k \otimes \sigma^2_{k+1}$ and $\sigma^3_k \otimes \sigma^3_{k+1}$ are equal. If two or all three of these coefficients are chosen differently, one finds the so-called XXZ or XYZ spin chains, respectively. Choosing one coefficient to be zero yields the XX or XY model.

space $\mathbb{V}_0$, i.e. on the product space $\mathbb{V}_k \otimes \mathbb{V}_0$. The defining relations for the Lax operator are given by an integrability equation similar to the Yang–Baxter equation for the R-matrix, the so-called *RLL-relations* defined on $\mathbb{V}_k \otimes \mathbb{V}_0 \otimes \mathbb{V}_{\bar{0}}$:

$$\mathrm{R}_{0\bar{0}}(u-v)\mathrm{L}_{k0}(u)\mathrm{L}_{k\bar{0}}(v) = \mathrm{L}_{k\bar{0}}(v)\mathrm{L}_{k0}(u)\mathrm{R}_{0\bar{0}}(u-v). \tag{5.9}$$

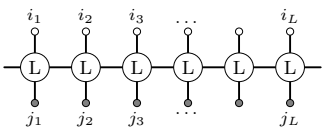
Resembling the definition of a Lie algebra via commutators, the RLL-relations relate the two products $\mathrm{L}_{k0}(u)\mathrm{L}_{k\bar{0}}(v)$ and $\mathrm{L}_{k\bar{0}}(v)\mathrm{L}_{k0}(u)$ to each other and can thus be understood as a generalized commutation relation defining L. Here the R-matrix acts as an intertwiner on two auxiliary spaces $\mathbb{V}_0 \otimes \mathbb{V}_{\bar{0}}$ labeled 0 and $\bar{0}$. Since we are interested in integrable models, we assume that the R-matrix obeys the quantum Yang–Baxter equation (3.5). Alternatively, given the Lax operator, we may understand (5.9) as a defining equation for the R-matrix, which for consistency has to obey the quantum Yang–Baxter equation. In fact, if the auxiliary space $\mathbb{V}_0$ and the physical space $\mathbb{V}_k$ are the same, the Lax operator is often identified with the R-matrix (up to convenient shifts in the spectral parameter and overall scalar factors). In particular, for fundamental models such as the Heisenberg spin chain, where the physical and auxiliary spaces carry the same representation, one often finds

$$\mathrm{L}_{k0}(u) \simeq \mathrm{R}_{k0}(u - \tfrac{i}{2}). \tag{5.10}$$

The full power of the Lax formalism comes into play when the physical and auxiliary representations are different. Note that in principle nothing prevents us from choosing arbitrary representations on the physical and auxiliary spaces and to study possible solutions to the Yang–Baxter or RLL-equations.

One of the most important quantities in the context of integrable spin chains is the *monodromy matrix* $\mathrm{T}^{\alpha}{}_{\beta}(u)$ defined as a product of L of the above Lax matrices:[48]

$$\mathrm{T}^{i_1\dots i_L,\alpha}_{j_1\dots j_L,\beta}(u) = \mathrm{L}^{\alpha i_1}_{k_2 j_1}(u-u_1)\mathrm{L}^{k_2 i_2}_{k_3 j_2}(u-u_2)\dots\mathrm{L}^{k_L i_L}_{\beta j_L}(u-u_L) =$$



. (5.11)

Alternatively we write this monodromy in the often more user-friendly form

$$\mathrm{T}_{1,\dots,L,0}(u) = \mathrm{L}_{10}(u-u_1)\mathrm{L}_{20}(u-u_2)\dots\mathrm{L}_{L0}(u-u_L). \tag{5.12}$$

In general, one may consider inhomogeneous spin chains with non-trivial parameters u_k for $k = 1,\dots L$. In the following we will restrict to homogeneous chains with inhomogeneities $u_k = 0$.

The monodromy $\mathrm{T}_0 \equiv \mathrm{T}_{1,\dots,L,0}$ acts on L physical and one auxiliary space and obeys the same equation as the underlying Lax matrix, namely the *RTT-relation* that we have already encountered (3.94):[49]

$$\mathrm{R}_{0\bar{0}}(u-v)\mathrm{T}_0(u)\mathrm{T}_{\bar{0}}(v) = \mathrm{T}_{\bar{0}}(v)\mathrm{T}_0(u)\mathrm{R}_{0\bar{0}}(u-v). \tag{5.13}$$

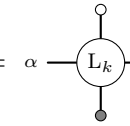
As we have seen in Section 3.3.2, the Yangian algebra may be defined via this equation. In order to identify the generators of the Yangian algebra (in the first realization) within this formalism, we will now expand the above monodromy around the point $u = \infty$. For this purpose we first

[48] This is the discretized version of the path-ordered exponential (2.48).

[49] For a more extended pedagogical introduction to the Lax formalism and algebraic Bethe ansatz see e.g. [71,72].

require a more explicit expression for the Lax operator, which is typically written as (up to overall factors):

$$(\mathrm{L}_{k0})^{\alpha}{}_{\beta}(u) = -iu\,\mathbb{1}_k \otimes (\mathbb{1}_0)^{\alpha}{}_{\beta} + i\mathrm{J}^a_k \otimes (E^a_0)^{\alpha}{}_{\beta} \equiv -iu[\mathbb{1}]^{\alpha}{}_{\beta} + i[\mathrm{J}_k]^{\alpha}{}_{\beta} = \quad . \tag{5.14}$$



Here E^a_0 denotes a generator in the fundamental representation of the underlying symmetry algebra $\mathfrak{g}$ acting on the auxiliary space and J^a may correspond to a generator in a different representation. For convenience we suppress the fundamental indices in the physical space $\mathbb{V}_k$ and, as usual, we sum over the adjoint index a.[50] Thus, the monodromy matrix on a chain ranging from 1 to L takes the form

$$\mathrm{T}^{\alpha}{}_{\beta}(u) = (-iu)^L\Big([\mathbb{1}_1]^{\alpha}{}_{\gamma_2} - \tfrac{1}{u}\,[\mathrm{J}_1]^{\alpha}{}_{\gamma_2}\Big)\ldots\Big([\mathbb{1}_L]^{\gamma_L}{}_{\beta} - \tfrac{1}{u}\,[\mathrm{J}_L]^{\gamma_L}{}_{\beta}\Big), \tag{5.15}$$

such that

$$\mathrm{T}^{\alpha}{}_{\beta}(u) = (-iu)^L\bigg([\mathbb{1}]^{\alpha}{}_{\beta} - \frac{1}{u}\sum_{k=1}^{L}[\mathrm{J}_k]^{\alpha}{}_{\beta} + \frac{1}{u^2}\sum_{k=1}^{L}\sum_{j=1}^{k-1}[\mathrm{J}_j]^{\alpha}{}_{\gamma}\,[\mathrm{J}_k]^{\gamma}{}_{\beta} + \mathcal{O}\Big(\tfrac{1}{u^3}\Big)\bigg). \tag{5.16}$$

If we now consider the example of $E_a = \sigma_a$, i.e. the fundamental representation of $\mathfrak{su}(2)$, we find

$$[\mathrm{J}_k]^{\alpha}{}_{\gamma}\,[\mathrm{J}_\ell]^{\gamma}{}_{\beta} = \mathrm{J}^b_k \otimes (E^b_0)^{\alpha}{}_{\gamma}\,\mathrm{J}^c_\ell \otimes (E^c_0)^{\gamma}{}_{\beta} = i\epsilon_{abc}\mathrm{J}^b_k\mathrm{J}^c_\ell \otimes (E^a_0)^{\alpha}{}_{\beta} + \mathrm{J}^b_k\mathrm{J}^c_\ell\delta_{bc} \otimes (\mathbb{1}_0)^{\alpha}{}_{\beta}, \tag{5.17}$$

where we have used that for the Pauli matrices $\sigma_b\sigma_c = i\epsilon_{bca}\sigma_a + \delta_{bc}$. Hence, we obtain[51]

$$\begin{aligned}\mathrm{T}^{\alpha}{}_{\beta}(u) = (-iu)^L\bigg[-\tfrac{1}{u}\sum_{k=1}^{L}\mathrm{J}^a_k + \tfrac{i}{u^2}\,\epsilon_{abc}\sum_{k=1}^{L}\sum_{j=1}^{k-1}\mathrm{J}^b_j\,\mathrm{J}^c_k\bigg] \otimes (E^a_0)^{\alpha}{}_{\beta}\\ +(-iu)^L\bigg[\mathbb{1} + \tfrac{1}{u^2}\sum_{k=1}^{L}\sum_{j=1}^{k-1}\mathrm{J}^b_j\,\mathrm{J}^b_k\bigg] \otimes (\mathbb{1}_0)^{\alpha}{}_{\beta} + (-iu)^L\mathcal{O}\Big(\tfrac{1}{u^3}\Big).\end{aligned} \tag{5.18}$$

The first bracket gives rise to the Yangian level-zero and level-one generators at orders $\frac{1}{u}$ and $\frac{1}{u^2}$, respectively (here with $f_{abc} = \epsilon_{abc}$):

$$\mathrm{J}_a = \sum_{k=1}^{L}\mathrm{J}_{k,a}, \qquad \widehat{\mathrm{J}}_a = f_{abc}\sum_{k=1}^{L}\sum_{j=1}^{k-1}\mathrm{J}_{b,j}\,\mathrm{J}_{c,k}\ . \tag{5.19}$$

The second line of (5.18) represents a linear combination of the identity and higher powers of the level-zero charges, which are less interesting for algebraic considerations.[52] Higher orders in the $\frac{1}{u}$-expansion contain higher-level Yangian generators as well as powers of the lower-level generators.

[50] Here we have chosen our convention such that $\mathrm{L}(\frac{i}{2}) = \mathbb{P}$ for $\mathrm{J}^a = \frac{\sigma^a}{2i}$ and $E^a = \sigma^a$, with σ^a denoting the Pauli matrices.

[51] See also [63] for a similar discussion.

[52] We use

$$\sum_{k=1}^{L}\sum_{j=1}^{k-1}\mathrm{J}^a_j\,\mathrm{J}^a_k = \tfrac{1}{2}\sum_{k=1}^{L}\sum_{j=1}^{L}\mathrm{J}^a_j\,\mathrm{J}^a_k - \tfrac{1}{2}\sum_k\mathrm{J}^a_k\mathrm{J}^a_k = \tfrac{1}{2}\mathrm{J}^a\mathrm{J}^a - \tfrac{1}{2}\sum_k\mathrm{J}^a_k\mathrm{J}^a_k\ , \tag{5.20}$$

where typically $\mathrm{J}^a_k\mathrm{J}^a_k \simeq \mathbb{1}_k$.

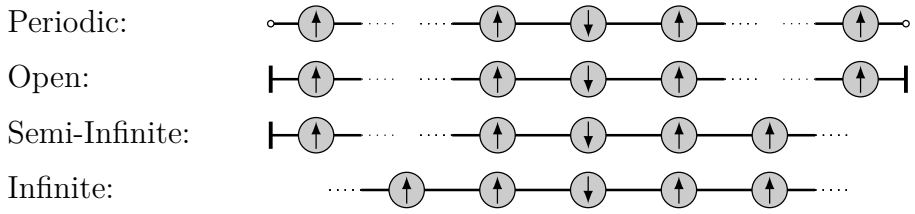


Figure 7: We distinguish between periodic ($\circ$) and open ($\dashv$) as well as semi-infinite, i.e. half-open, and infinite spin chains.

Note that the above expansion of the spin chain monodromy is very similar to the continuous version in (2.51). In fact, we could have guessed these expressions for the Yangian generators by simply replacing $\int \mathrm{d}x \to \sum_k$. Here we have no local contribution to the level-one generators coming from the homogeneous monodromy matrix. We may obtain an additional local contribution $\sum_k u_k \mathrm{J}^a_k$ by introducing nontrivial inhomogeneities $u_k \neq 0$ in (5.11).

In principal, checking the Yangian symmetry of a spin chain model, i.e. of a defining Hamiltonian, provides an integrability test. In general, however, the question of whether a Hamiltonian has exact Yangian symmetry strongly depends on the boundary conditions of the underlying model.

5.2 Different Boundary Conditions

Here we briefly comment on the compatibility of the Yangian with different spin chain boundary conditions. Similar considerations apply to the case of continuous two-dimensional field theories discussed above. We will see that an exact Yangian symmetry is generically not compatible with finite boundary conditions. This means that the Hamiltonian defining the model does typically not commute with the Yangian level-one generators. Nevertheless the Yangian symmetry can give nontrivial constraints in the bulk of the system, i.e. the symmetry equation is obeyed modulo boundary terms. While in this case the spectrum is not organized in Yangian multiplets, one may use Yangian symmetry to bootstrap a Hamiltonian, cf. e.g. [73]. Suppose we make an ansatz for a Hamiltonian H, then the equation

$$[\widehat{\mathrm{J}}, \mathrm{H}] = \text{boundary terms} \tag{5.21}$$

yields non-trivial constraints on the bulk part of this Hamiltonian and in this sense represents a non-trivial symmetry of the model. In certain special cases, one may even find an exact Yangian symmetry as indicated below.

Periodic Boundary Conditions. Periodic boundary conditions are implemented by identifying the spin chain site $L+1$ with the site 1, such that the sites L and 1 are nearest neighbors. Generically, exact Yangian symmetry is not compatible with periodic boundary conditions. This can be seen from the ordered structure of the level-one symmetry in the first realization. Hence, to define a Yangian level-one generator for a periodic system, one has to choose an origin on the periodic chain such that the sites can be considered as being ordered according to their relative position to this origin. When applied to a periodic quantity, this typically implies boundary terms that spoil an invariance equation.

As an example consider again the Heisenberg or XXX Hamiltonian introduced above:

$$\mathrm{H} \equiv \mathrm{H}_{\mathrm{XXX}} = \sum_{k=1}^{L} (\mathbb{1}_{k,k+1} - \mathbb{P}_{k,k+1}). \tag{5.22}$$

The $\mathfrak{su}(2)$ symmetry of the model means that

$$[\mathrm{J}_a, \mathrm{H}] = 0, \tag{5.23}$$

with $\mathrm{J}_a = \sum_{k=1}^{L} \frac{\sigma_{a,k}}{2i}$ denoting the Lie algebra (or level-zero) generators of $\mathfrak{su}(2)$. Here again $\sigma_{a=1,2,3}$ are the Pauli matrices and $[\mathrm{J}_a, \mathrm{J}_b] = \epsilon_{abc}\mathrm{J}_c$. The corresponding Yangian $Y[\mathfrak{su}(2)]$ is spanned by these level-zero generators and the level-one generators

$$\widehat{\mathrm{J}}_a = \epsilon_{abc} \sum_{1\leq j<k\leq L} \mathrm{J}_{b,j}\mathrm{J}_{c,k}. \tag{5.24}$$

Importantly, here we have chosen a spin chain origin at site 1 or L, respectively. While this point is not distinguished by the periodic Hamiltonian (5.22), defining the level-one generator (5.24) requires this choice. We remember from the $\mathfrak{su}(2)$ example discussed above that $[\epsilon_{abc}\mathrm{J}_b \otimes \mathrm{J}_c, \mathbb{P}] = \mathrm{J}_a \otimes \mathbb{1} - \mathbb{1} \otimes \mathrm{J}_a$ which implies that

$$[\widehat{\mathrm{J}}_a, \mathrm{H}] = \mathrm{J}_{a,1} - \mathrm{J}_{a,L} \equiv \mathrm{J}_a \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} - \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathrm{J}_a. \tag{5.25}$$

Here the dots stand for $L-3$ identity operators $\mathbb{1}$. Hence, the Heisenberg Hamiltonian commutes with the level-one symmetry up to boundary terms, i.e. terms that act only on the boundary of the spin chain. Similar considerations apply to the Yangian symmetry of the spin chain Hamiltonian of $\mathcal{N} = 4$ super Yang–Mills theory discussed in Section 6.2 [20]. Note that particular examples of spin chain models exist, whose periodic boundary conditions are compatible with an exact Yangian symmetry, see [74].

Cyclic Boundary Conditions. In contrast to periodic boundary conditions which only imply that the sites L and 1 are neighbors, cyclic boundary conditions in addition require that the system is invariant under cyclic shifts $k \to k+1$ or $k \to k-1$ of the sites. This implies that the total momentum of all spin chain excitations has to be zero. In particular, this means that the Yangian generators should commute with the shift operator U, which induces cyclic permutations by one site. That this yields additional constraints can be seen from evaluating [22, 75]

$$[\mathrm{U}, \widehat{\mathrm{J}}_a] = \mathrm{U}\, f_{abc}\Big[\sum_{k=1}^{L}\sum_{j=1}^{k-1} \mathrm{J}_{b,j}\mathrm{J}_{c,k} - \sum_{k=2}^{L+1}\sum_{j=2}^{k-1} \mathrm{J}_{b,j}\mathrm{J}_{c,k}\Big] = \mathrm{U}\left(f_{abc}f_{dbc}\mathrm{J}_{d,1} - 2f_{abc}\mathrm{J}_{b,1}\mathrm{J}_c\right). \tag{5.26}$$

In general, the right hand side of this equation does not vanish, which emphasizes the dependence of the level-one generators on the choice of an origin of the chain. The first term is proportional to the dual coxeter number $\mathfrak{c}_2 = f_{abc}f_{bcd}$, which vanishes only for some particular algebras $\mathfrak{g}_{\mathfrak{c}_2=0}$. The second term is proportional to the level-zero generator J_c and is generically non-zero.

Suppose, however, we forget about Hamiltonians for the moment and instead consider Yangian invariants, i.e. 'states' $|I\rangle$ that are annihilated by the Yangian generators. Then at least for certain algebras $\mathfrak{g}_{\mathfrak{c}_2=0}$ one may define cyclic invariants $|I\rangle$ of the Yangian algebra:[53]

$$\widehat{\mathrm{J}}\,|I\rangle = 0, \qquad \text{consistent if} \qquad \mathrm{J}\,|I\rangle = 0, \quad \text{and} \quad \mathfrak{c}_2 = 0. \tag{5.27}$$

These conditions imply that the right hand side of (5.26) vanishes on $|I\rangle$. This type of cyclic boundary conditions is particularly interesting since it applies to scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory discussed in Section 6 with $\mathfrak{g}_{\mathfrak{c}_2=0} = \mathfrak{psu}(2,2|4)$.

[53] See [76] for a pedagogical introduction to a systematic study of Yangian invariants.

Open and Semi-infinite Boundary Conditions. Similar to the case of periodic boundary conditions, also open boundaries are often not compatible with a full Yangian symmetry. Here the situation very much depends on the specific bulk and boundary part of the Hamiltonian:

$$\mathrm{H} = \mathrm{H}_{\text{bulk}} + \mathrm{H}_{\text{left-boundary}} + \mathrm{H}_{\text{right-boundary}}. \tag{5.28}$$

Analogous considerations apply to semi-infinite (=half-open) boundaries where either the left or right boundary is absent and the chain extends to infinity on that side.

Suppose we have a system with a level-zero symmetry algebra $\mathfrak{g}$ in the bulk and a symmetry $\mathfrak{h}$ at the boundary (see e.g. [77]) such that $(\mathfrak{g}, \mathfrak{h})$ form a symmetric pair, which means that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}. \tag{5.29}$$

Furthermore we assume to have a Yangian symmetry $Y[\mathfrak{g}]$ in the bulk. Then the whole system, including the boundary, often still has a *twisted Yangian* symmetry $Y[\mathfrak{g}, \mathfrak{h}]$ whose level-zero generators are J_i (note the index i), while the level-one generators have the modified form (note the index p)

$$\tilde{\mathrm{J}}_p = \widehat{\mathrm{J}}_p + \tfrac{1}{2} f_{pqi} \mathrm{J}_i \mathrm{J}_q, = \widehat{\mathrm{J}}_p + \tfrac{1}{4} [\mathcal{C}_{\mathfrak{h}}, \mathrm{J}_p], \tag{5.30}$$

with $\mathcal{C}_{\mathfrak{h}}$ representing the quadratic Casimir of $\mathfrak{g}$ restricted to $\mathfrak{h}$. Importantly, the indices $i, j, \ldots$ correspond to generators of $\mathfrak{h}$ and the indices $p, q, \ldots$ to generators of $\mathfrak{m}$. The index sums thus only run over subsets of all index values. For further details and references see for instance [78].

Infinite Chain: No Boundaries. As seen above, the Yangian generators are most naturally defined with boundaries at $\pm\infty$, i.e. with no boundaries at all. On such an infinite chain, the right hand side of (5.25) vanishes and the Hamiltonian has exact Yangian symmetry. However, typically such infinite spin chain systems are rather of formal interest and do not directly represent physical models. In particular, their spectrum is not quantized due to the underlying noncompact space.

5.3 Periodic Chains, Transfer Matrix and Local Charges

In the case of periodic spin chain boundary conditions, the so-called *transfer matrix* $\mathfrak{t}(u) \equiv \mathfrak{t}_{1\ldots L}(u)$ is defined as the trace over the monodromy matrix (the trace implements the periodicity)

$$\mathfrak{t}^{i_1 \ldots i_L}_{j_1 \ldots j_L}(u) = \operatorname{Tr} \mathrm{T}^{i_1 \ldots i_L, \alpha}_{j_1 \ldots j_L, \beta}(u) \equiv \mathfrak{t}^{i_1 \ldots i_L, \alpha}_{j_1 \ldots j_L, \alpha}(u), \tag{5.31}$$

and acts on the whole spin chain:

$$\mathfrak{t}(u) : \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L \to \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L. \tag{5.32}$$

In our alternative notation we may write this trace over (5.11) as

$$\mathfrak{t}(u) = \operatorname{Tr}_0 \mathrm{L}_{10}(u) \mathrm{L}_{20}(u) \ldots \mathrm{L}_{L0}(u), \tag{5.33}$$

where we suppress the free indices and only indicate the vector spaces $\mathbb{V}_k$ on which the Lax-matrices act. The index 0 denotes the auxiliary space which is traced, e.g. $\operatorname{Tr}_0 (\mathrm{L}_{10})^{i_1}_{j_1} (\mathrm{L}_{20})^{i_2}_{j_2} = \mathrm{L}^{\ell i_1}_{k j_1} \mathrm{L}^{k i_2}_{\ell j_2}$. It follows from the RTT-relations (5.13) that this transfer matrix commutes with itself when evaluated at different spectral parameters (see e.g. [72, 79]):

$$[\mathfrak{t}(u), \mathfrak{t}(v)] = 0. \tag{5.34}$$

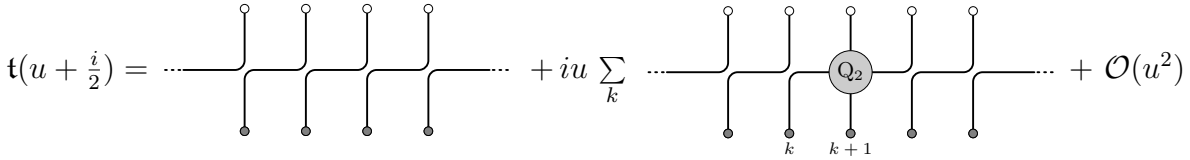


Figure 8: Expanding the transfer matrix around $u = \frac{i}{2}$ gives rise to the shift operator U at zeroth order, to the shift operator times the Hamiltonian $\mathrm{U}\mathcal{Q}_2$ at order u, and an infinite set of additional commuting charges at higher orders of the spectral parameter u.

Let us furthermore assume that the considered system forms part of the so-called *fundamental models*, for which the auxiliary and quantum spaces are the same (or isomorphic) and for which a special point $u = \frac{i}{2}$ exists such that $\mathrm{L}(\frac{i}{2}) = \mathbb{P}$, cf. [80]. This is the case for simple examples of integrable spin chains such as the Heisenberg model. Hence the point $u = \frac{i}{2}$ is dinstinguished and we expand the transfer matrix around this point in order to obtain u-independent conserved charge operators (cf. Figure 8)

$$\mathfrak{t}(u+\tfrac{i}{2}) = \mathrm{U}\exp i\sum_{r=2}^{L} u^{r-1}\mathcal{Q}_r = \mathrm{U} + iu\,\mathrm{U}\mathcal{Q}_2 + u^2\,(i\,\mathrm{U}\mathcal{Q}_3 - \tfrac{1}{2}\,\mathrm{U}\mathcal{Q}_2\mathcal{Q}_2) + \mathcal{O}(u^3), \tag{5.35}$$

Here $\mathrm{U} = \mathfrak{t}(\frac{i}{2})$ denotes the shift operator that is given by a product of iterative permutations $\mathbb{P}_{k,k+1}$ and induces a cyclic permutation of all spin chain sites, e.g.

$$\mathrm{U}\,|X_1, X_2 \ldots, X_L\rangle = |X_2, \ldots, X_L, X_1\rangle\,. \tag{5.36}$$

Since we may pull out the overall factor U in (5.35), the $\mathcal{Q}_r$ denote local operators on the spin chain. It follows from (5.34) that all operators $\mathcal{Q}_r$ mutually commute such that the transfer matrix furnishes a generating functional for local integrable charge operators [81, 80]:

$$\mathcal{Q}_r = -\frac{i}{(r-1)!}\frac{\mathrm{d}}{\mathrm{d}u^{r-1}}\log\mathfrak{t}(u)\big|_{u=\frac{i}{2}}\,, \qquad r \geq 2. \tag{5.37}$$

Remember that the first of these charges $\mathcal{Q}_2 = \mathrm{H}$ is typically the Hamiltonian that is chosen to define the model's dynamics. The logarithmic derivative of the transfer matrix generates $L-1$ mutually commuting charges and thereby naturally associates the interaction range of the longest operator to the dimension of the transfer matrix. Note that taking the logarithmic derivative ensures that the charges are local [82] and that we have divided the definition of the charges by the common shift operator

$$\mathrm{U} = \mathfrak{t}(\tfrac{i}{2}) = e^{i\mathcal{Q}_1}. \tag{5.38}$$

Hence, the first two powers in the expansion of the transfer matrix define the momentum operator and the Hamiltonian of the model (cf. Figure 8):

$$\mathcal{Q}_1 = -i\log\mathfrak{t}(\tfrac{i}{2}), \qquad\qquad \mathcal{Q}_2 = -i\frac{\mathrm{d}}{\mathrm{d}u}\log\mathfrak{t}(u)\Big|_{u=\frac{i}{2}}. \tag{5.39}$$

Expressing the momentum operator $\mathcal{Q}_1$ as the logarithm of an operator only formally illustrates the analogy to the other charges. In particular, (5.39) implies that the density of the Hamiltonian is proportional to the logarithmic derivative of the Lax operator:

$$\mathrm{H}_{k,k+1} = \mathcal{Q}_{2,k,k+1} \simeq \frac{\mathrm{d}}{\mathrm{d}u}\log\mathrm{L}_{k,k+1}(u)\Big|_{u=\frac{i}{2}}. \tag{5.40}$$

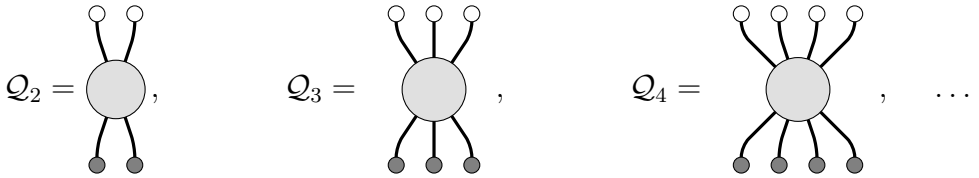


Figure 9: The interaction range of the charge $\mathcal{Q}_n$ is n, i.e. the charge operator acts on n neighboring sites at the same time.

Note that this expression makes only sense for fundamental models with $\mathbb{V}_j \simeq \mathbb{V}_0$ where the Lax operator L_{k0} acts on the same space as the Hamiltonian density $\mathrm{H}_{k,k+1}$, namely on $\mathbb{V}_k \otimes \mathbb{V}_0 \simeq \mathbb{V}_k \otimes \mathbb{V}_{k+1}$.

In general, i.e. for *non-fundamental models*, the Yangian generators and the local Hamiltonians do not have to originate from the same monodromy. If the auxiliary and physical spaces are not identical, one may impose the following version of the RLL-relation (5.9) (now exchanging the roles of auxiliary and quantum spaces) defined on $\mathbb{V}_k \otimes \mathbb{V}_{k+1} \otimes \mathbb{V}_0$:

$$\mathrm{R}_{k,k+1}(v)\mathrm{L}_{k0}(u+v)\mathrm{L}_{k+1,0}(u) = \mathrm{L}_{k+1,0}(u)\mathrm{L}_{k,0}(u+v)\mathrm{R}_{k,k+1}(v). \tag{5.41}$$

This equation follows from the quantum Yang–Baxter equation (3.5) by the identification $\mathrm{R}_{12} \to \mathrm{R}_{k,k+1}$, $\mathrm{R}_{13} \to \mathrm{L}_{k0}$ and $\mathrm{R}_{23} \to \mathrm{L}_{k+1,0}$. Given a Lax operator L_{k0}, this RLL-relation defines an R-matrix on $\mathbb{V}_k \otimes \mathbb{V}_{k+1}$ and allows to define a Hamiltonian via its logarithmic derivative (cf. [80]):

$$\mathrm{H}_{k,k+1} = \mathcal{Q}_{2k,k+1} = i\frac{\mathrm{d}}{\mathrm{d}u}\log \mathrm{R}_{k,k+1}(u)\Big|_{u=0}. \tag{5.42}$$

Higher local charges can be obtained from the expansion of a transfer matrix of the form (5.33) with the replacement $\mathrm{L} \to \mathrm{R}$ and the trace taken over a quantum space.

Example. In the case of the Heisenberg spin chain the first few local charges take the form,[54] cf. Figure 9

$$\begin{aligned}
&\mathcal{Q}_2 = [1] - [2,1],\\
&\mathcal{Q}_3 = \tfrac{i}{2}([3,1,2] - [2,3,1]),\\
&\mathcal{Q}_4 = \tfrac{1}{3}(-[1] + 2[2,1] - [3,2,1] + [2,3,4,1] - [2,4,1,3] - [3,1,4,2] + [4,1,2,3]),
\end{aligned} \tag{5.43}$$

where we have used the notation of (5.7) to display the permutations which furnish the charges' building blocks. We note that on a periodic spin chain we have $[1,2] \equiv [1]$ since the difference of these two operators only acts nontrivially on boundaries, which are absent on a periodic chain.

5.4 Master Symmetry and Boost Automorphism

In this subsection we briefly demonstrate the role of the discrete version of the Lorentz boost.

[54] In order to obtain exactly these expressions for the charges (which annihilate the ferromagnetic vaccuum state) from the above formalism, one should modify the definition of the Lax matrix (5.14) by an overall function of u. This modification does not change the physics of the model. We also note that the $\mathfrak{gl}(N)$ symmetric spin chain has the same charges.

Boost automorphism and monodromy matrix. Let us derive an interesting relation for the monodromy matrix of an ordinary integrable short-range spin chain along the lines of the original paper by Tetel'man [14]. First of all we introduce the Hamiltonian density using the logarithmic derivative of an R-matrix with $\mathrm{R}(0) = \mathbb{P}$:

$$\mathrm{H}_{k,k+1} = i\,\mathbb{P}_{k,k+1}\,\dot{\mathrm{R}}_{k,k+1}(0), \qquad \dot{\mathrm{R}}_{k,k+1}(0) = \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{R}_{k,k+1}(u)\Big|_{u=0}. \tag{5.44}$$

This density acts on the spin chain sites k and $k+1$, i.e. on $\mathbb{V}_k \otimes \mathbb{V}_{k+1}$. We consider the RLL-relation in the form of (5.41):

$$\mathrm{R}_{12}(v)\mathrm{L}_{10}(u+v)\mathrm{L}_{20}(u) = \mathrm{L}_{20}(u)\mathrm{L}_{10}(u+v)\mathrm{R}_{12}(v). \tag{5.45}$$

Differentiating with respect to v and setting v to zero afterwards yields the following differential equation for the Lax-matrix:

$$\dot{\mathrm{L}}_{10}(u)\mathrm{L}_{20}(u) - \mathrm{L}_{10}(u)\dot{\mathrm{L}}_{20}(u) = i[\mathrm{H}_{12}, \mathrm{L}_{10}(u)\mathrm{L}_{20}(u)]. \tag{5.46}$$

Here we used that the R-matrix obeys the initial condition $\mathrm{R}_{12}(0) = \mathbb{P}_{12}$. Now we rename $1 \to k$ and $2 \to k+1$ and multiply this equation from the left by $\prod_{1=j}^{k-1}\mathrm{L}_{0j}$ and from the right by $\prod_{\ell=k+1}^{L}\mathrm{L}_{0\ell}$. This yields

$$\Big(\prod_{j=1}^{k-1}\mathrm{L}_{0j}\Big)\dot{\mathrm{L}}_{0k}\Big(\prod_{\ell=k+1}^{L}\mathrm{L}_{0\ell}\Big) - \Big(\prod_{j=1}^{k}\mathrm{L}_{0j}\Big)\dot{\mathrm{L}}_{0,k+1}\Big(\prod_{\ell=k+2}^{L}\mathrm{L}_{0\ell}\Big) = i\Big[\mathrm{H}_{k,k+1}, \prod_{j=1}^{L}\mathrm{L}_{0j}\Big]. \tag{5.47}$$

If we now furthermore multiply this equation by k and sum over k from 1 to L , this yields

$$\frac{\mathrm{dT}(u)}{\mathrm{d}u} + 0 \times \dot{\mathrm{L}}_{01}\Big(\prod_{j=2}^{L}\mathrm{L}_{0j}\Big) - L \times \Big(\prod_{j=1}^{L}\mathrm{L}_{0j}\Big)\dot{\mathrm{L}}_{0,L+1} = i\Big[\sum_{k=1}^{L} k\,\mathrm{H}_{k,k+1}, \mathrm{T}(u)\Big], \tag{5.48}$$

where $\mathrm{T}(u)$ denotes the monodromy matrix (5.11) with suppressed indices. Note that the appearence of only right boundary terms originates in our choice of labeling the first and last spin chain leg 1 and L.

If we take the limit of an *infinite spin chain*, i.e. $-\infty \leftarrow 1$ and $L \to \infty$, the boundary terms on the left hand side of (5.48) drop out and we find the equation

$$\frac{\mathrm{dT}(u)}{\mathrm{d}u} = i[\mathcal{B}, \mathrm{T}(u)], \tag{5.49}$$

where we have defined the *spin chain boost operator*

$$\mathcal{B} = \sum_{k=-\infty}^{\infty} k\,\mathrm{H}_{k,k+1}. \tag{5.50}$$

Note that this is the discrete version of the field theory boost (2.19), obtained by replacing $\int dx\, x \to \sum_k k$. Using the expansion (5.18) of the monodromy matrix, we see that this equation implies the relation

$$[\mathcal{B}, \widehat{\mathrm{J}}_a] \simeq \mathrm{J}_a. \tag{5.51}$$

Hence, the spin chain boost operator defined above corresponds to Drinfel'd's boost automorphism of the Yangian algebra (3.24).

Master symmetry and transfer matrix. The concept of *master symmetry* of integrable models dates back to 1981 [83]. It denotes a symmetry whose iterative application to the constants of motion leaves their commutator invariant. Consequently, the master symmetry maps integrable charges to integrable charges and thereby generates a set of infinitely many commuting operators.

In order to identify such a symmetry we now take the trace Tr_0 over the monodromy matrix in (5.49) to obtain the following equation for the transfer matrix

$$\frac{\mathrm{d}\mathfrak{t}(u)}{\mathrm{d}u} = i[\mathcal{B}, \mathfrak{t}(u)]. \tag{5.52}$$

We assume that the expansion of the transfer matrix $\mathfrak{t}(u)$ yields the local charges via (5.37). Hence, (5.52) is a remarkable statement since it implies

$$\mathcal{Q}_2 = \mathrm{U}^{-1}[\mathcal{B}, \mathrm{U}] = \mathrm{U}^{-1}\mathcal{B}\,\mathrm{U} - \mathcal{B}, \tag{5.53}$$

$$\mathcal{Q}_3 = +\tfrac{i}{2}\mathrm{U}^{-1}[\mathcal{B}, \mathrm{U}\mathcal{Q}_2] - \tfrac{i}{2}\mathcal{Q}_2\mathcal{Q}_2 = \tfrac{i}{2}[\mathcal{B}, \mathcal{Q}_2], \tag{5.54}$$

$$\mathcal{Q}_4 = \tfrac{i}{3}[\mathcal{B}, \mathcal{Q}_3], \tag{5.55}$$

$$\dots$$

The spin chain boost operator $\mathcal{B} = \mathcal{B}[\mathcal{Q}_2]$, the first moment of the Hamiltonian, therefore recursively generates the algebra of local integrable charges

$$\mathcal{Q}_{r+1} = \frac{i}{r}\,[\mathcal{B}, \mathcal{Q}_r]. \tag{5.56}$$

It represents a master symmetry of the short-range integrable model. Note that the charges $\mathcal{Q}_3$ and $\mathcal{Q}_4$ given in (5.43) may be obtained using the boost operator in this way.[55]

Poincaré algebra. Let us compare the algebra spanned by the local charges and the boost operator to the ordinary two-dimensional Poincaré algebra, cf. [15]:

$$[\mathrm{P}, \mathrm{H}] = 0, \qquad [\mathrm{B}, \mathrm{P}] = \mathrm{H}, \qquad [\mathrm{B}, \mathrm{H}] = \mathrm{P}. \tag{5.57}$$

Here space and time translations P and H are 'rotated' into each other by the Lorentz boost B. For our integrable spin chain the Poincaré algebra is enhanced according to

$$\mathrm{H} \to \mathcal{Q}_2, \mathcal{Q}_3, \dots, \qquad \mathrm{P} \to \mathcal{Q}_1 \qquad \mathrm{B} \to \mathcal{B}, \tag{5.58}$$

and we have the commutation relations

$$\begin{aligned} &[\mathcal{Q}_1, \mathcal{Q}_2] = 0 \qquad [\mathcal{B}, \mathcal{Q}_1] \simeq \mathcal{Q}_2 \qquad [\mathcal{B}, \mathcal{Q}_2] \simeq \mathcal{Q}_3 \qquad [\mathcal{B}, \mathcal{Q}_3] \simeq \mathcal{Q}_4 \qquad \dots \\ &[\mathcal{Q}_2, \mathcal{Q}_3] = 0 \\ &[\mathcal{Q}_3, \mathcal{Q}_4] = 0 \\ &\quad \dots \end{aligned} \tag{5.59}$$

Space and time translation are supplemented by an infinite set of symmetries. The Lorentz boost translating between the two symmetries P and H of the ordinary Poincaré algebra (5.58), here takes the role of a ladder operator. For the integrable spin chain the sequence of symmetries does not close and we find a tower of conserved charge operators, cf. Figure 10.[56] Mapping conserved charges to conserved charges, the boost thus represents a master symmetry in the above sense.

[55] Notably, this method produces additional boundary terms which vanish on infinite or periodic spin chains and can therefore be dropped.

[56] Also in continuous field theories one may construct a tower of local charges, which are mapped onto themselves by the Lorentz boost, see e.g. [84]

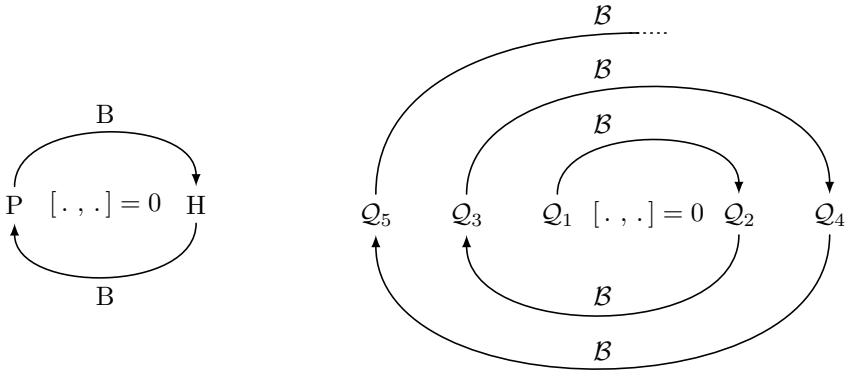


Figure 10: Analogy between the Poincaré algebra with generators B, P and H and the algebra of the boost and the integrable charges.

Periodic Chain. To get rid of the boundary terms in (5.48), we have taken the limit of an infinite spin chain. On a periodic spin chain, the definition of the boost operator is obsolete since it crucially depends on the choice of a spin chain origin. Also in this respect, the boost operator resembles the Yangian level-one generators. Nevertheless, the short-range charge operators of the infinite spin chain are the same as those of the periodic chain since in both cases boundary operators vanish. Hence, while only properly defined on an infinite chain, the boost operator can be formally used to generate the set of periodic integrable charges. In fact, if the boost operator were well-defined on periodic chains, the finite transformation corresponding to (5.52) would merely constitute a similarity transformation.

5.5 More Boosts and Long-Range Spin Chains

We have seen above that integrable spin chains feature a tower of commuting charges $\mathcal{Q}_n$, $n = 1, 2, \ldots$. Furthermore, we have seen that the boost operator

$$\mathcal{B} = \sum_k k\, \mathrm{H}_k = \sum_k k\, \mathcal{Q}_{2,k}, \tag{5.60}$$

plays the role of a master symmetry. Certainly, we may also define higher 'boost operators' associated with the higher local charges:

$$\mathcal{B}[\mathcal{Q}_n] = \sum_k k\, \mathcal{Q}_{n,k}. \tag{5.61}$$

Here $\mathcal{B} = \mathcal{B}[\mathcal{Q}_2]$ denotes the boost operator encountered in the previous subsections. A natural question is, which role the higher boosts play for the considered spin chain models. We will see that they allow to define an integrable (so-called *long-range*) deformation of the above *short-range* spin chain models, which plays an important role in the context of the gauge/gravity duality further discussed below (cf. Section 6.2).

In fact, one may define deformed charge operators $\mathcal{Q}_n(\lambda)$ via the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\, \mathcal{Q}_n(\lambda) = i\alpha_k(\lambda)\Big[\mathcal{B}[\mathcal{Q}_k(\lambda)], \mathcal{Q}_n(\lambda)\Big], \qquad k > 2. \tag{5.62}$$

Here we sum over k and $\alpha_k(\lambda)$ denotes some function of λ whose form specifies the precise deformation. If the charges $\mathcal{Q}_n(0)$ commute among each other, also the solutions $\mathcal{Q}_n(\lambda)$ to the

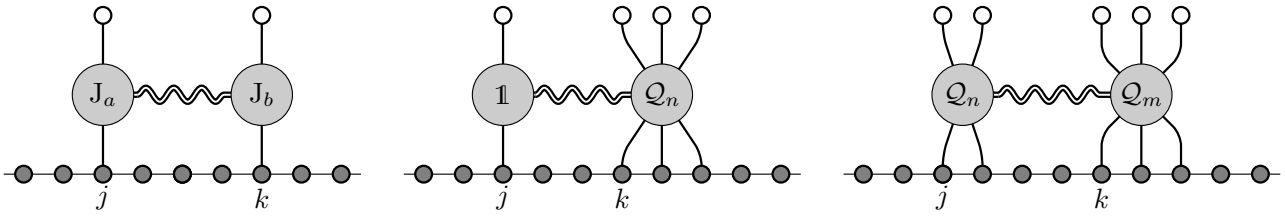


Figure 11: Overview over bilocal operators: Yangian level-one generators $\widehat{\mathrm{J}}$, boost operators $\mathcal{B}[\mathcal{Q}_n] \equiv [\mathbb{1}\,|\mathcal{Q}_n]$ and bilocal charges $[\mathcal{Q}_n|\mathcal{Q}_m]$.

defining equation (5.62) commute. Hence, the deformed charges $\mathcal{Q}_n(\lambda)$ define an integrable system. The deformation of the local charges typically increases their interaction range on the spin chain. Suppose we denote the local short-range charges of the previous subsections by $\mathcal{Q}_n \equiv \mathcal{Q}_n(0) \equiv \mathcal{Q}_n^{(0)}$. Then the charges $\mathcal{Q}_n(\lambda)$ define a long-range spin chain model, i.e. a model whose defining Hamiltonian and higher integrable charges have longer and longer range of interaction when going to higher powers of the parameter λ. Again, the long-range Hamiltonian $\mathrm{H}(\lambda) = \mathcal{Q}_2(\lambda)$ defines the dynamics of the model and the parameter λ may be understood as a coupling constant.

In general, perturbatively long-ranged spin chains may be defined as deformations of the above short-range chains, e.g. the Heisenberg chain. The short-range charges $\mathcal{Q}_n$ are taken to be the leading order $\mathcal{Q}_n^{(0)}$ in a power series

$$\mathcal{Q}_n(\lambda) = \mathcal{Q}_n^{(0)} + \lambda\mathcal{Q}_n^{(1)} + \lambda^2\mathcal{Q}_n^{(2)} + \mathcal{O}(\lambda^3), \qquad \mathcal{Q}_n^{(0)} = \mathcal{Q}_n, \tag{5.63}$$

such that the interaction range of the charges grows with the perturbative order in λ. The long-range charges can still be written as linear combinations of local and homogeneous operators O_k

$$\mathcal{Q}_r(\lambda) = \sum_k c_{r,k}(\lambda)\mathrm{O}_k, \tag{5.64}$$

but now with coefficients $c_{r,k}(\lambda)$ which are formal power series in λ starting at a certain order. The charges have to obey the integrability condition $[\mathcal{Q}_r(\lambda), \mathcal{Q}_s(\lambda)] = 0$ order by order in λ, which is guaranteed by the deformation equation (5.62).

To make connection to the Yangian algebra, we note that also the Yangian generators should be deformed in order to preserve their commutation relations with the local (long-range) Hamiltonians. By assumption, the level-zero generators J_a commute with the charges $\mathcal{Q}_n$ and thus also with the boost operators $\mathcal{B}[\mathcal{Q}_n]$. For the level-one generators one uses a deformation equation analogous to (5.62):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\widehat{\mathrm{J}}_a(\lambda) = i\alpha_k(\lambda)\Big[\mathcal{B}[\mathcal{Q}_k(\lambda)], \widehat{\mathrm{J}}_a(\lambda)\Big]. \tag{5.65}$$

This equation defines the long-range level-one Yangian generators. In fact, we may remember that in analogy to the level-one symmetry, also the boost can be understood as a formally bilocal expression, i.e. $\mathcal{B}[\mathcal{Q}_n] = [\mathbb{1}\,|\mathcal{Q}_n]$ (see (2.20) for the continuous case). Taking this consideration further one can define even more general bilocal charge operators, cf. Figure 11:

$$[\mathcal{Q}_n|\mathcal{Q}_m] = \sum_{j<k} \mathcal{Q}_{n,j}\mathcal{Q}_{m,k}. \tag{5.66}$$

These may also be employed to generate long-range deformations of spin chain Hamiltonians via similar differential equations as (5.62) and (5.65). We note that these bilocal charge generators

induce so-called dressing phase contributions to the dilatation operator of $\mathcal{N}=4$ super Yang–Mills theory, cf. Section 6. However, discussing these in more detail is beyond the scope of these lectures; see [16] for further elaboration.[57]

Example. To illustrate the deformation of level-one generators via (5.65) let us once more consider $\mathfrak{g}=\mathfrak{su}(2)$ with generators $\mathrm{J}_a=\frac{\sigma_a}{2i}$ in the fundamental representation. We choose $\alpha_3=1$ and $\alpha_{k>3}=0$, i.e. we deform the level-one generators only using the higher boost $\mathcal{B}[\mathcal{Q}_3]$. The first higher charge $\mathcal{Q}_3$ of the $\mathfrak{su}(2)$ Heisenberg chain is given by (5.43) and we can plug it into the definition of the generalized boost operator (5.61). Then we find the following leading-order deformation of the level-one generator:

$$\widehat{\mathrm{J}}_a(\lambda)=\widehat{\mathrm{J}}_a(0)+\lambda\, i[\mathcal{B}[\mathcal{Q}_3],\widehat{\mathrm{J}}_a]+\mathcal{O}(\lambda^2)=\epsilon_{abc}\sum_{j<k}\mathrm{J}_{b,j}\mathrm{J}_{c,k}+\lambda\,\epsilon_{abc}\sum_{k}\mathrm{J}_{b,k}\mathrm{J}_{c,k+1}+\mathcal{O}(\lambda^2). \quad (5.67)$$

Here the level-zero generators $\mathrm{J}_a(\lambda)\equiv\mathrm{J}_a$ remain undeformed. Notably, this deformation of the level-one generators of the Heisenberg model corresponds to the two-loop deformation of the Yangian symmetry of the $\mathfrak{su}(2)$ dilatation operator of $\mathcal{N}=4$ super Yang–Mills theory, cf. [74, 88, 73]. In this context the parameter λ represents the 't Hooft coupling constant, cf. Section 6.

6 Yangian Symmetry in 4d Field Theory

> *"The models analyzed in this paper, formulated in two space-time dimensions, are clearly unrealistic. However, we believe that the phenomenon exhibited by these models is indicative of what one would expect in more realistic models. In fact the restriction to two dimensions is only in order to have an asymptotically free theory in which one has an explicit expansion parameter (N). The only asymptotically free theory in four dimensions necessarily involves gauge fields and does not lend itself to any simple approximation."* D. Gross and A. Neveu 1974 [13]

Let us see that in fact Yangian symmetry can also be found in dimensions greater than two. In order to get a glimpse on how this happens, we first have to introduce a very interesting model, the maximally supersymmetric Yang–Mills theory in four dimensions. For a selection of helpful reviews see for instance [89].

6.1 $\mathcal{N}=4$ Super Yang–Mills Theory

Four-dimensional $\mathcal{N}=4$ super Yang–Mills (SYM) theory was originally introduced in the 1980s as the dimensional reduction of a ten dimensional super Yang–Mills theory with fermions [90, 91]. Compactification on a six dimensional torus gives rise to a four-dimensional field content comprised of a gauge field $\mathcal{A}_\mu$, four Dirac spinors Ψ_a, $\bar{\Psi}^a$ as well as six scalars Φ_k, $k=1,\dots,6$. Furthermore we have an adjoint covariant derivative $\mathcal{D}_\mu=\partial_\mu-ig[\mathcal{A}_\mu,\ \cdot\ ]$. The Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}}=\mathrm{Tr}\Big[&\tfrac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\tfrac{1}{2}\mathcal{D}^\mu\Phi^n\mathcal{D}_\mu\Phi_n+\bar{\Psi}^a_{\dot\alpha}\sigma_\mu^{\dot\alpha\beta}\mathcal{D}^\mu\Psi_{\beta a}-\tfrac{1}{4}g^2[\Phi^m,\Phi^n][\Phi_m,\Phi_n]\\&-\tfrac{1}{2}ig\Psi_{a\alpha}\Sigma^{ab}_m\varepsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta b}]-\tfrac{1}{2}ig\bar{\Psi}^a_{\dot\alpha}\Sigma^m_{ab}\varepsilon^{\dot\alpha\dot\beta}[\Phi_m,\bar{\Psi}^b_{\dot\beta}]\Big],\end{aligned} \quad (6.1)$$

[57] Note that this construction of long-range spin chains using boost operators may also be generalized to chains of trigonometric type [85] or to chains with open boundary conditions [86]. Furthermore interesting relations to (inhomogeneous versions of) Baxter's corner transfer matrix exist [87].

with a field strength defined by $\mathcal{F}_{\mu\nu} = ig^{-1}[\mathcal{D}_\mu, \mathcal{D}_\nu]$.[58] The four- and six- dimensional sigma matrices obey the relations

$$\{\sigma^\mu, \sigma^\nu\} = \eta^{\mu\nu}, \qquad \{\Sigma^m, \Sigma^n\} = \eta^{mn}. \tag{6.2}$$

All fields transform in the adjoint representation of a semisimple Lie group. In what follows we take this gauge group to be SU(N). The $\mathcal{N} = 4$ SYM theory action is uniquely determined up to two free paramters; the dimensionless coupling constant $g_{\rm YM}$ and the rank of the gauge group N.[59]

Symmetry. The different indices appearing in the Lagrangian represent different symmetries. Contraction of all indices shows the symmetry invariance of the respective terms:

- Spacetime symmetry: The indices μ, ν correspond to the vector representation of the Lorentz group. Indices α, β and $\dot\alpha, \dot\beta$ represent the left and right handed spinor representations of the Lorentz group.
- Global internal symmetry (R-symmetry): The Lagrangian has a global internal SO(6) $\simeq$ SU(4) symmetry acting on the SO(6) vector indices $m, n = 1, \ldots, 6$.
- Local internal symmetry (Gauge Symmetry): The Lagrangian has a local SU(N) gauge symmetry. Above the respective indices are hidden in the trace Tr.

In fact, the Poincaré and R-symmetry of the Lagrangian are enhanced to *superconformal symmetry*. To be precise, the action of this four-dimensional quantum field theory is invariant under the set of generators[60]

$$\{\mathrm{L}, \bar{\mathrm{L}}, \mathrm{P}, \mathrm{K}, \mathrm{D}, \mathrm{R} | \mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{S}, \bar{\mathrm{S}}\} \quad \in \quad \mathfrak{psu}(2,2|4), \tag{6.3}$$

which span the $\mathcal{N} = 4$ superconformal algebra divided by its center, cf. e.g. [92]. The above generators correspond to the set of Lie algebra (level-zero) generators J that we encountered in the previous sections. They satisfy the graded (due to the fermionic supersymmetry generators) commutation relations[61]

$$[\mathrm{J}_a, \mathrm{J}_b\} = f_{ab}{}^c \, \mathrm{J}_c. \tag{6.4}$$

This Lie algebra contains the Poincaré Lorentz- and momentum generators L ,$\bar{\mathrm{L}}$ and P as well as the momentum supercharges Q and $\bar{\mathrm{Q}}$. Being conformal, the symmetry algebra also encloses the conformal boost K, the dilatation generator D and the conformal supercharges S and $\bar{\mathrm{S}}$; all fields of the theory are massless. The action is invariant under an $\mathfrak{su}(4)$ internal symmetry contained in $\mathfrak{psu}(2,2|4)$ and identified with the generators R. This R-symmetry rotates the supercharges into each other.[62]

One of the most remarkable features of $\mathcal{N} = 4$ SYM theory is the fact that its coupling constant is constant, i.e. independent of the renormalization scale μ, cf. [94–96]:

$$\beta = \mu \frac{\partial g}{\partial \mu} = 0. \tag{6.5}$$

[58] Here, spacetime indices are denoted by $\mu, \nu, .. = 1, .., 4$ while spinor indices of $\mathfrak{su}(2)$ given by $\alpha, \beta, ..$ or $\dot\alpha, \dot\beta, ..$ take two values. Vector and spinor indices of $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ are denoted by $m, n, ..$ and $a, b, ..$ and range from 1 to 6 or 1 to 4, respectively.

[59] Here we ignore a topological term $\sim \theta \, \mathcal{F}\tilde{\mathcal{F}}$.

[60] Note that the Q's here denote supercharges and should not be confused with the local charges $\mathcal{Q}_n$ discussed in the previous section.

[61] In this section we distinguish between upper and lower adjoint indices.

[62] This form of supersymmetry with generators transforming non-trivially under the internal R-symmetry was referred to as *hypersymmetry* in the original work on $\mathcal{N} = 4$ SYM theory [93, 90].

This scale independence implies that (super)conformal symmetry is preserved at the quantum level making this model the paradigm of four-dimensional quantum field theories. In fact, this large amount of symmetry is further enhanced in the so-called planar limit, which turns this gauge theory into an integrable model.

Planar Limit. We consider the action of $\mathcal{N} = 4$ SYM theory in terms of the standard Yang–Mills coupling g_{YM}:

$$S_{\mathrm{YM}} = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x \, \mathcal{L}_{\mathrm{YM}}(g = 1). \tag{6.6}$$

For the large-N limit it proves useful to redefine the free parameters of the theory by introducing the so-called 't Hooft coupling [97]

$$\lambda = g_{\mathrm{YM}}^2 N. \tag{6.7}$$

The limit $N \to \infty$, $\lambda =$ fixed is called the *'t Hooft*, *large-N* or *planar limit*. The latter name stems from the fact that in this limit only Feynman diagrams contribute that can be drawn on a plane (as opposed to different topologies). Taking N to infinity and restricting to the leading perturbative order, the 't Hooft coupling λ is the essential expansion parameter in the planar limit. It is related to the coupling constant g in (6.1) by $\lambda = 8\pi^2 g^2$.

The above large-N limit was originally introduced in 1973 by 't Hooft who investigated U(N) gauge theories with regard to inseparable quark bound states as found in QCD [97]. The 't Hooft limit was his approach to simplifying the strong coupling behavior of QCD. He also noticed that in this limit the expansion of correlators very much resembles the genus expansion in a string theory with coupling $g_{\mathrm{s}} = 1/N$. Later this became manifest in form of the AdS/CFT correspondence, c.f. Section 6.4.

In the planar limit $\mathcal{N} = 4$ SYM theory acquires an additional symmetry, namely *integrability* which leads to many simplifications in explicit calculations and which is realized on several types of observables in the form of Yangian symmetry (see below).

Gauge invariant operators and spin chain picture. The prime observables of a conformal field theory are correlators of gauge invariant local operators. In fact, the knowledge of all two point-functions is equivalent to knowing the spectrum of the theory, while the three-point functions encode the conformal structure constants and thus the dynamics. Hence, it is important to understand how these local operators look like.

All fields of $\mathcal{N} = 4$ SYM theory transform in the adjoint representation of the gauge group SU(N). Thus, we can associate a fundamental and an anti-fundamental color index to each of the fields $X^{\mathfrak{a}}{}_{\mathfrak{b}} \in \{\mathcal{D}_\mu, \Psi_{\alpha a}, \bar{\Psi}^a_{\dot{\alpha}}, \Phi_k, \mathcal{F}_{\mu\nu}\}^{\mathfrak{a}}{}_{\mathfrak{b}}$. A gauge transformation acts as $X \mapsto UXU^{-1}$. Taking color traces of products of fields transforming homogeneously under gauge transformations, we can thus construct gauge invariant local operators as:

$$\mathcal{O}(x) = \mathrm{Tr}[X_1 X_2 \dots X_L](x) = (X_1)^{\mathfrak{a}_1}{}_{\mathfrak{a}_2} (X_2)^{\mathfrak{a}_2}{}_{\mathfrak{a}_3} \dots (X_L)^{\mathfrak{a}_L}{}_{\mathfrak{a}_1}(x). \tag{6.8}$$

Here all fields X_i are understood to be evaluated at the same spacetime point x^μ.[63] Since all gauge indices are contracted, it is clear that operators of this form are gauge singlets. In the following we will refer to the trace operators (6.8) also as local gauge invariant *states* and make use of the identification

$$\mathrm{Tr}[X_1 \dots X_L] \equiv |X_1 \dots X_L\rangle \, . \tag{6.9}$$

[63]Note that the gauge field $\mathcal{A}_\mu$ cannot be used for the construction of such states because it transforms inhomogeneously under gauge transformations.

Obviously, also products of traces will be gauge invariant operators. However, in the strict[64] large-N limit these are not relevant for the correlators we are interested in. The set of states of the form (6.8) thus forms a (cyclic) basis for local gauge invariant states in $\mathcal{N}=4$ SYM theory at large N.

The above trace operators can be considered as a tensor product of fields transforming under a representation of the theory's symmetry algebra plus additional cyclic boundary conditions. In other words we may call the above basis states *spin chains* with cyclic boundary conditions, e.g.

$$\operatorname{Tr} \Phi_1 \,\ldots\, \Phi_1\, \Phi_2\, \Phi_1 \,\ldots\, \Phi_1. \tag{6.10}$$

In particular, the representation of the superconformal symmetry takes the tensor product form

$$\mathrm{J}_a = \sum_{k=1}^{L} \mathrm{J}_{a,k}, \qquad \mathrm{J}_a \in \mathfrak{psu}(2,2|4), \tag{6.11}$$

where $\mathrm{J}_{a,k}$ denotes the representation on one of the fields. Except for the Lorentz- and internal rotations L, $\bar{\mathrm{L}}$ and R, the representations of all symmetry generators of $\mathcal{N}=4$ SYM theory acquire radiative corrections in the coupling constant when promoted to higher loop orders

$$\mathrm{J}_a(g) = \mathrm{J}_a^{(0)} + g\,\mathrm{J}_a^{(1)} + g^2\,\mathrm{J}_a^{(2)} + \ldots, \qquad \mathrm{J}_a \in \mathfrak{psu}(2,2|4). \tag{6.12}$$

The graded commutation relations of $\mathfrak{psu}(2,2|4)$ with structure constants $f_{ab}{}^c$ are not affected by these deformations

$$[\mathrm{J}_a(g), \mathrm{J}_b(g)\} = f_{ab}{}^c\, \mathrm{J}_c(g). \tag{6.13}$$

In what follows we will refer to the perturbative order $g^{2\ell}$ as the ℓ-loop order.

6.2 Dilatation Operator

In this section we briefly indicate, how Yangian symmetry acts on the dilatation operator alias the Hamiltonian of $\mathcal{N}=4$ super Yang–Mills theory. In fact, studying eigenvalues of the dilatation operator is equivalent to studying the energy spectrum of this theory. This can be seen in a particular radial quantization scheme (cf. e.g. [98]), where the dilatation operator generates time shifts

$$\mathrm{D} = -ir\frac{\partial}{\partial r} = -i\frac{\partial}{\partial t}. \tag{6.14}$$

Hence, studying conformal dimensions Δ of gauge invariant states $\mathcal{O}$ with $\mathrm{D}\mathcal{O} = \Delta\mathcal{O}$ is equivalent to the study of the energy spectrum of these states. Thus, one often does not distinguish between the terms energy spectrum and anomalous dimensions, and the dilatation operator is referred to as the Hamiltonian of the theory.

Integrable structures in the $\mathfrak{su}(2)$ subsector at one loop. As indicated above, solving the spectral problem of local operators in $\mathcal{N}=4$ SYM theory reduces to the problem of finding the spectrum of the dilatation operator D. Furthermore we have seen that a basis for gauge invariant local operators is given by traces of the form (6.8):

$$\mathcal{O} = \operatorname{Tr}[X_1 X_2 \ldots X_L]. \tag{6.15}$$

[64] If we speak of the *strict* large N limit here, we mean that only the leading order in $1/N$ is kept.

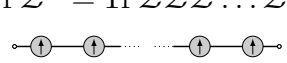
Therefore it suggests itself to think about diagonalizing $\mathrm{D} = \mathrm{D}_0 + \delta\mathrm{D}(g)$ on this basis set of states.

Diagonalizing the anomalous part of the dilatation operator $\delta\mathrm{D}(g)$ is a very challenging task even at one-loop order. Thus, it appears reasonable to think of dividing the problem into smaller pieces, i.e. to diagonalize the dilatation generator on a subset of local gauge invariant states. All such subsectors closed under the action of the dilatation generator (i.e. closed under renormalization) were identified in [99]. As a result one finds several closed sectors, each characterized by its field content and the residual symmetry. For this classification of subsectors, it proves useful to combine the six scalar fields Φ_i of $\mathcal{N} = 4$ SYM theory into complex fields

$$\mathcal{X} = \Phi_1 + i\Phi_2, \quad \mathcal{W} = \Phi_3 + i\Phi_4, \quad \mathcal{Z} = \Phi_5 + i\Phi_6, \tag{6.16}$$

and their conjugates. The simplest subsector is then given by the (half-BPS) states of the form

$$\mathrm{Tr}\, \mathcal{Z}^L = \mathrm{Tr}\, \mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\mathcal{Z},$$



(6.17)

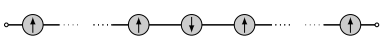
where the picture again emphasizes the analogy between trace operators and spin chains. We will refer to these states as the cyclic vacuum states of length L. The name vacuum already indicates the relation to the ferromagnetic vacuum of a spin chain. The vacuum has the residual symmetry[65]

$$\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \times \mathfrak{u}(1)^3, \tag{6.18}$$

where all generators except for the length measuring operator act trivially. That is, the length L is the only non-vanishing quantum number characterizing the cyclic vacuum state. Hence, diagonalizing the anomalous dilatation generator on these states is trivial.

In order to proceed to the simplest *non-trivial* subsector, we excite also the field $\mathcal{W}$ to consider states of the form

$$\mathrm{Tr}\, \mathcal{Z}\ldots\mathcal{Z}\mathcal{W}\mathcal{Z}\ldots\mathcal{Z}$$



(6.19)

The residual symmetry of this sector is

$$\mathfrak{su}(2) \times \mathfrak{u}(1)^2, \tag{6.20}$$

and it is therefore referred to as the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM theory. The fields $\mathcal{Z}$ and $\mathcal{W}$ transform under the fundamental representation of $\mathfrak{su}(2)$ and the $\mathfrak{u}(1)$ charges are given by the length operator and the anomalous dimension $\delta\mathrm{D}$. The resulting states can be identified with spin chains built out of fundamental modules of $\mathfrak{su}(2)$, i.e. with chains of the Heisenberg type encountered above.

The one-loop dilatation operator in the $\mathfrak{su}(2)$ subsector can be found by explicitly renormalizing the involved Feynman diagrams and reading off the renormalization constants. At one loop order Minahan and Zarembo found the following expression in their famous paper [17]:[66]

$$\delta\mathrm{D}_{\mathfrak{su}(2)}(g) = \frac{g^2}{2}\sum_{k=1}^{L}(\mathbb{1}_k - \mathbb{P}_{k,k+1}) + \mathcal{O}(g^3) = \frac{g^2}{2}([1] - [2,1]). \tag{6.21}$$

[65]Note that there is no known vacuum state with residual symmetry $\mathfrak{psu}(2,2|4)$ which would appear natural in $\mathcal{N} = 4$ SYM theory.

[66]Note that including all scalar fields of $\mathcal{N} = 4$ SYM theory leads to the $\mathfrak{so}(6)$ subsector which is closed only at one loop order in perturbation theory. In the original work [17], however, one-loop integrability was shown in the whole scalar $\mathfrak{so}(6)$ sector including the $\mathfrak{su}(2)$ sector.

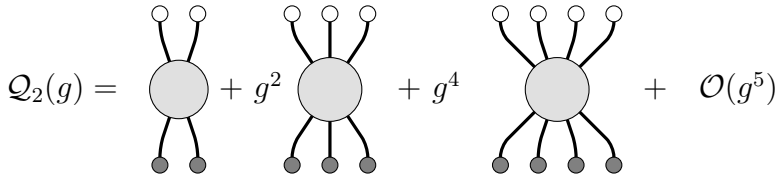


Figure 12: The interaction range of the integrable long-range charges grows with increasing order of the coupling constant.

Here $\mathbb{1}_k$ and $\mathbb{P}_{k,k+1}$ denote the identity and permutation operator acting on sites k and $k+1$, respectively. Due to the periodicity of the trace states on which this operator acts, we identify the sites $L+1$ and 1. Remarkably, the one-loop anomalous dilatation generator in (6.21) equals the Hamiltonian of the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ spin chain (with cyclic boundary conditions), the prime example of an integrable model. Its discovery was a huge breakthrough in the study of $\mathcal{N}=4$ SYM theory.

Higher Loop Integrability: Long-Range Integrable Spin Chains. It was shown in [18] that the integrable structures of the spectral problem in $\mathcal{N}=4$ SYM theory extend to higher loop orders. Noting the existence of degeneracies in the spectrum at two-loop order, it was possible to construct perturbative corrections to the first two integrable charges in the $\mathfrak{su}(2)$ subsector

$$\mathcal{Q}_2(g) = \mathcal{Q}_2^{(0)} + g^2\mathcal{Q}_2^{(1)} + \mathcal{O}(g^3), \qquad \mathcal{Q}_3(g) = \mathcal{Q}_3^{(0)} + g^2\mathcal{Q}_3^{(1)} + \mathcal{O}(g^3), \tag{6.22}$$

such that these operators commute

$$[\mathcal{Q}_2^{(0)}, \mathcal{Q}_3^{(0)}] + g^2\Big([\mathcal{Q}_2^{(1)}, \mathcal{Q}_3^{(0)}] + [\mathcal{Q}_2^{(0)}, \mathcal{Q}_3^{(1)}]\Big) + \mathcal{O}(g^3) = 0. \tag{6.23}$$

Importantly, as opposed to the one-loop level, these deformed charges are only *perturbatively integrable*, i.e. they commute up to higher powers in the coupling constant.

Also the interaction range of the higher charge orders increases with the power of the coupling constant, e.g. the two loop correction to the dilatation generator acts on three sites at the same time, see also Figure 12:

$$\mathcal{Q}^{(1)}_{2,\mathfrak{su}(2)} = -3[1] + 4[2,1] - [3,2,1]. \tag{6.24}$$

In fact, this property is expected with regard to Feynman diagram calculations. The more powers of the coupling contribute to a given perturbative order, the more fields can be involved into the interactions.

This is exactly the type of long-range spin chain that we considered in Section 5.5. We note that in fact the higher perturbative orders of the dilatation operator can be generated by the generalized boost operators discussed above [16]. The corresponding two-loop deformation of the Yangian level-one generators was given in (5.67).

Yangian symmetry of the dilatation operator. Having identified the one-loop dilatation operator with the Heisenberg spin chain Hamiltonian, the above example (5.25) immediately shows that the Yangian algebra $Y[\mathfrak{su}(2)]$ commutes with this operator up to boundary terms. As demonstrated by Dolan, Nappi and Witten [20], this property extends to the complete one-loop dilatation operator of [99] which commutes with the level-one generators of $Y[\mathfrak{psu}(2,2|4)]$ into boundary terms:

$$[\widehat{\mathrm{J}}_a, \delta\mathrm{D}^{(1)}] \simeq \mathrm{J}_{a,1} - \mathrm{J}_{a,L}. \tag{6.25}$$

In order to extend this relation to higher loops one major difficulty is that not even the two-loop dilatation operator is known for the complete theory, i.e. on the full $\mathfrak{psu}(2,2|4)$ spin chain. However, the (asymptotic) higher loop dilatation operator is known in certain subsectors and some statements on the Yangian symmetry can be made, see e.g. [100, 16]. In the $\mathfrak{su}(2)$ sector for instance, the representation of the Yangian generators may be deformed using the boost and bilocal charges of Section 5.5.

6.3 Scattering Amplitudes

In this section we briefly indicate how Yangian symmetry is realized on the scattering matrix of $\mathcal{N}=4$ SYM theory.

Four Dimensional Kinematics. We are interested in scattering of n massless fields in $\mathcal{N}=4$ SYM theory. Therefore it is useful to express the n external four-momenta p_k^μ as bi-spinors $p_k^{\alpha\dot\alpha} = (\sigma_\mu)^{\alpha\dot\alpha} p_k^\mu$ and explicitly solve the on-shell condition $p_k^2 = 0$ for all external particles in terms of commuting spinors [101]

$$p_k^{\alpha\dot\alpha} = \lambda_k^\alpha \bar\lambda_k^{\dot\alpha}, \quad \text{for} \quad k = 1, \dots, n. \tag{6.26}$$

Here λ_k^α and $\bar\lambda_k^{\dot\alpha}$ are complex conjugate bosonic Lorentz-spinors with indices $\alpha, \beta, \dots = 1, 2$ and $\dot\alpha, \dot\beta, \dots = 1, 2$. The spinor decomposition (6.26) of massless momenta in four dimensions is unique only up to a complex rescaling

$$\lambda^\alpha \to c\lambda^\alpha, \qquad \bar\lambda^{\dot\alpha} \to c^{-1}\bar\lambda^{\dot\alpha}. \tag{6.27}$$

All physical quantities should therefore be independent of this transformation.[67]

It is straightforward to construct invariants under Lorentz symmetry out of the momentum spinors according to

$$\langle ij\rangle := \langle \lambda_i \lambda_j\rangle = \varepsilon_{\alpha\beta}\lambda_i^\alpha \lambda_j^\beta, \qquad [ij] := [\bar\lambda_i \bar\lambda_j] = \varepsilon_{\dot\alpha\dot\beta}\bar\lambda_i^{\dot\alpha}\bar\lambda_j^{\dot\beta}. \tag{6.28}$$

These spinor brackets furnish fundamental building blocks for the construction of scattering amplitudes as we will see below.

Color Ordering. Tree-level scattering amplitudes in SU(N) $\mathcal{N}=4$ SYM theory can be expanded according to

$$\hat A_n(\{\lambda_i, h_i, a_i\}) = \sum_{\sigma\in S_n/\mathbb{Z}_n} A_n(\lambda_{\sigma(1)}, h_{\sigma(1)}, \dots, \lambda_{\sigma(n)}, h_{\sigma(n)}) \operatorname{Tr} \mathrm{T}^{a_{\sigma(1)}}\mathrm{T}^{a_{\sigma(2)}}\dots\mathrm{T}^{a_{\sigma(n)}}, \tag{6.29}$$

such that the amplitude's color structure is encoded in traces over gauge group generators T^a of $\mathfrak{su}(N)$.[68] Here the symbol h_i denotes the helicity of the ith particle. This straightforward separation of color and kinematical structure allows to reduce the non-trivial scattering problem to the kinematical part of the amplitude A_n. The cyclicity of the trace implies that this kinematical scattering amplitude A_n is a function invariant under cyclic permutations of its arguments.

[67] Physical scattering amplitudes require at least two negative energy particles. This is due to the two constraining equations $\sum_{k=1}^n p_k = 0$ and $p_n^2 = \left(\sum_{k=1}^{n-1} p_k\right)^2 = 0$. Here we will focus on positive energy solutions and consider all particles as incoming in what follows. The arguments in this chapter generalize to the inclusion of negative energy particles which, however, results in less clear expressions, cf. [23].

[68] At higher loop orders and at the same time going beyond the planar limit also multi-trace contributions have to be added to this expansion. Here we will be interested in the planar tree level where only single traces appear.

Superfield. In order to compute scattering amplitudes in $\mathcal{N}=4$ SYM theory it is most convenient to make use of the fact that fields with different helicity transform in different representations of the internal R-symmetry. We may thus introduce fermionic spinors η^A, $A,B,\ldots=1,\ldots,4$, of $\mathfrak{su}(4)$ and collect all fields in a chiral on-shell superfield [95, 102]

$$\begin{aligned}\Phi(\lambda,\bar\lambda,\eta) =& G^+(\lambda,\bar\lambda)+\eta^A\Gamma_A(\lambda,\bar\lambda)+\tfrac{1}{2}\eta^A\eta^B S_{AB}(\lambda,\bar\lambda)\\ &+\tfrac{1}{3!}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\bar\Gamma^D(\lambda,\bar\lambda)+\tfrac{1}{4!}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D G^-(\lambda,\bar\lambda).\end{aligned}\tag{6.30}$$

Here each power of the Graßmann parameters η corresponds to a different representation of the R-symmetry. The on-shell gluons $G^\pm$, fermions $\Gamma/\bar\Gamma$, and scalars S have helicity ±1, $\pm\frac{1}{2}$ and 0.

Every analytic function of the superfield Φ can be expanded in terms of the Graßmann superspace coordinates η. The fields contributing to a certain order in this expansion are determined by the respective power in η. In particular, singlets of $\mathfrak{su}(4)$, i.e. symmetry invariant functions of the superfield, are proportional to $\eta^4=\frac{1}{4!}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D$. Considering the n-particle scattering amplitude as a superspace function

$$A_n(\Phi_1,\ldots,\Phi_n)=A_n(\Phi(\Lambda_1),\ldots,\Phi(\Lambda_n)),\qquad \Lambda_k=(\lambda_k,\bar\lambda_k,\eta_k)\tag{6.31}$$

we may thus expand it in terms of component amplitudes given by the coefficients of powers of η^4

$$A_n=\sum_{k=2}^{n-2}A_{n,k},\qquad\qquad \mathrm{B}A_{n,k}=4kA_{n,k}.\tag{6.32}$$

Here we have introduced the η counting generator $\mathrm{B}=\eta^A\partial/\partial\eta^A$ and used that supersymmetry implies $A_{n,1}=A_{n,n-1}=0$ [103]. In Minkowski signature the three-particle scattering amplitude A_3 of massless particles vanishes by kinematical arguments. Hence, the lowerst non-trivial amplitude is A_4, see Figure 13.

As seen in (6.32), scattering amplitudes in $\mathcal{N}=4$ SYM theory are categorized according to their helicity configuration $A_n=A_{n,2}+A_{n,3}+\ldots$. Remarkably, the so-called maximally helicity violating (MHV) amplitudes $A_n^{\mathrm{MHV}}=A_{n,2}$ can be written in a very compact fashion [104, 101]

$$A_n^{\mathrm{MHV}}=\frac{\delta^4(P)\,\delta^8(Q)}{\langle12\rangle\langle23\rangle\ldots\langle n1\rangle},\qquad P^{\alpha\dot\beta}=\sum_{k=1}^n\lambda_k^\alpha\bar\lambda_k^{\dot\beta},\quad Q^{\alpha B}=\sum_{k=1}^n\lambda_k^\alpha\eta_k^B,\tag{6.33}$$

with the Lorentz-invariant spinor brackets defined in (6.28). This definition of the amplitude ensures conservation of the overall momentum P and super-momentum Q.

Level-Zero Symmetry. Using spinor helicity superspace coordinates the one-particle representation of the superconformal algebra $\mathfrak{psu}(2,2|4)$ was written down by Witten [105]:

$$\begin{aligned}\mathrm{L}^a{}_b&=\lambda^a\partial_b-\tfrac{1}{2}\delta^a_b\lambda^c\partial_c, & \bar{\mathrm{L}}^{\dot\alpha}{}_{\dot\beta}&=\bar\lambda^{\dot\alpha}\bar\partial_{\dot\beta}-\tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\bar\lambda^{\dot\gamma}\bar\partial_{\dot\gamma},\\ \mathrm{D}&=\tfrac{1}{2}\partial_\gamma\lambda^\gamma+\tfrac{1}{2}\bar\lambda^{\dot\gamma}\bar\partial_{\dot\gamma}, & \mathrm{R}^A{}_B&=\eta^A\partial_B-\tfrac{1}{4}\delta^A_B\eta^C\partial_C,\\ \mathrm{Q}^{\alpha B}&=\lambda^\alpha\eta^B, & \mathrm{S}_{\alpha B}&=\partial_\alpha\partial_B,\\ \bar{\mathrm{Q}}^{\dot\alpha}_B&=\bar\lambda^{\dot\alpha}\partial_B, & \bar{\mathrm{S}}^B_{\dot\alpha}&=\eta^B\bar\partial_{\dot\alpha},\\ \mathrm{P}^{\alpha\dot\beta}&=\lambda^\alpha\bar\lambda^{\dot\beta}, & \mathrm{K}_{\alpha\dot\beta}&=\partial_\alpha\bar\partial_{\dot\beta},\end{aligned}\tag{6.34}$$

where we use the short-hand notation $\partial_\alpha=\partial/\partial\lambda^\alpha$, $\bar\partial_{\dot\alpha}=\partial/\partial\bar\lambda_{\dot\alpha}$ and $\partial_A=\partial/\partial\eta^A$. The above one-particle representation (6.34) is promoted to a representation on tree-level scattering amplitudes in $\mathcal{N}=4$ SYM theory by taking the tensor product, i.e. the primitive coproduct:

$$\mathrm{J}_a=\sum_{k=1}^n\mathrm{J}_{a,k}.\tag{6.35}$$

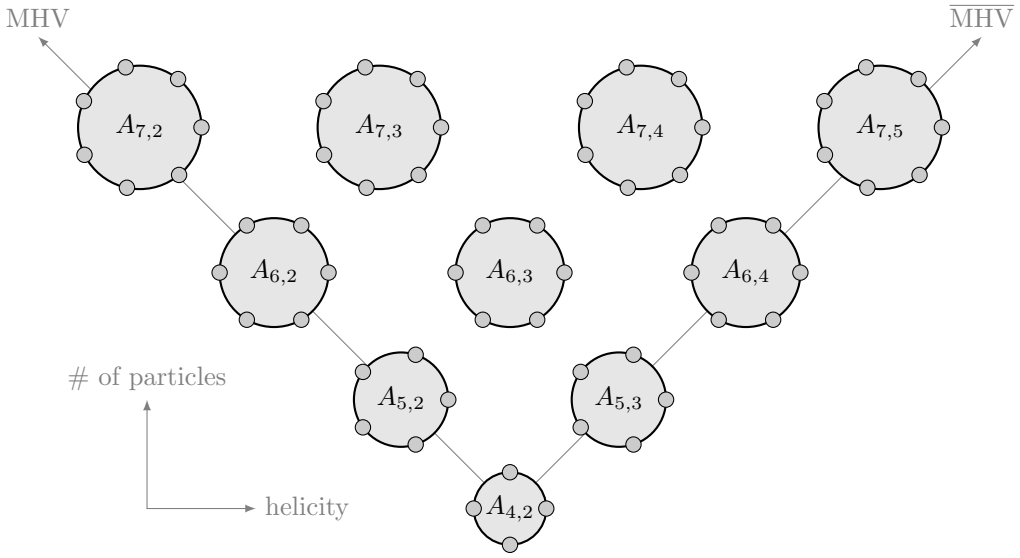


Figure 13: Scattering amplitudes $A_{n,k}$ in $\mathcal{N}=4$ SYM theory are nontrivial for $n>3$ external legs (and real momenta). They can be classified according to their helicity configuration measured by the respective power of the fermionic spinors η^{4k}. The simplest so-called MHV and $\overline{\text{MHV}}$ amplitudes are those on the left and right boundary of the above triangle.

Here $\mathrm{J}_{a,k}$ is the representation of the conformal symmetry generator J_a on the k-th leg $(\lambda_k, \bar{\lambda}_k, \eta_k)$ of A_n as specified in (6.34). Hence, the representation of the symmetry algebra on scattering amplitudes very much resembles the spin chain symmetry acting on local gauge invariant states (6.11). In fact, one may check that the scattering amplitude A_n is invariant under the action of the above generators [105]:[69]

$$\mathrm{J}_a A_n = 0. \tag{6.36}$$

Level-One Symmetry. We may define the level-one generators in the expected form

$$\widehat{\mathrm{J}}_a = f^a{}_{bc} \sum_{1\leq j<k\leq n} \mathrm{J}^b_j \, \mathrm{J}^c_k. \tag{6.37}$$

with a vanishing one-site representation as in (3.49). Due to the commutation relation (3.19) of the Yangian algebra given by

$$[\mathrm{J}_a, \mathrm{J}_b\} = f_{abc}\mathrm{J}^c, \qquad\qquad [\mathrm{J}_a, \widehat{\mathrm{J}}_b\} = f_{abc}\widehat{\mathrm{J}}^c, \tag{6.38}$$

it suffices to show the invariance of A_n under the level-zero symmetry and *one* level-one generator. This will imply invariance under the whole Yangian algebra via (6.38). For the explicit calculation it makes sense to choose the simplest level-one generator which is the level-one momentum operator $\widehat{\mathrm{P}}$ being linear in derivatives. This generator takes the explicit form

$$\widehat{\mathrm{P}}^{\alpha\dot{\alpha}} = \sum_{1\leq j<k\leq n} \Big[\mathrm{P}^{\gamma\dot{\gamma}}_j \Big(\mathrm{L}^{\alpha}_{k,\gamma}\delta^{\dot{\alpha}}_{\dot{\gamma}} + \bar{\mathrm{L}}^{\dot{\alpha}}_{k,\dot{\gamma}}\delta^{\alpha}_{\gamma} + \mathrm{D}_k \delta^{\alpha}_{\gamma}\delta^{\dot{\alpha}}_{\dot{\gamma}} \Big) + \mathrm{Q}^{\alpha C}_j \bar{\mathrm{Q}}^{\dot{\alpha}}_{k,C} - (j \leftrightarrow k) \Big]. \tag{6.39}$$

[69] Importantly, there are further corrections to the above expressions for the conformal level-zero generators in the limit where two external momenta of the amplitude become collinear. These limits require careful treatment and can also be tackled by algebraic considerations, see e.g. [23, 24, 106].

Here the $(j \leftrightarrow k)$ stands for all the previous terms under the sum but with j and k interchanged. Acting with the level-one momentum operator on the MHV amplitude, one finds (see [75] for more details)

$$\widehat{\mathrm{P}}^{\alpha\dot{\alpha}} A_n^{\mathrm{MHV}} = \frac{\lambda_1^\gamma \lambda_n^\beta + \lambda_1^\beta \lambda_n^\gamma}{\langle 1n \rangle} \epsilon_{\gamma\delta} \mathrm{P}^{\delta\dot{\alpha}} A_n^{\mathrm{MHV}} = 0. \tag{6.40}$$

Remember that due to (6.29) scattering amplitudes are invariant under cyclic shifts. Hence, we have a case of cyclic boundary conditions discussed in Section 5.2. Fortunately, the dual coxeter number $\mathfrak{c}_2$ of the symmetry algebra $\mathfrak{psu}(2,2|4)$ of $\mathcal{N}=4$ SYM theory is zero and scattering amplitudes are invariant under the level-zero symmetry (6.36) [22]. Thus, the case of (5.27) applies and we have a consistent realization of cyclic Yangian invariants.[70]

In fact, Yangian symmetry of the tree-level S-matrix of $\mathcal{N}=4$ SYM theory was first understood in the language of the so-called *dual conformal symmetry* [108, 22]. Furthermore, there is a map between all tree-level scattering amplitudes and certain contributions to the dilatation operator [109]. In particular, the four-point superamplitude furnishes the integral kernel for the one-loop dilatation operator [109, 110]. Hence, the Yangian symmetry of the four-point amplitude and the dilatation operator (both discussed above) can be shown to be consistent with each other [111].

4d versus 2d S-matrix. Let us finally compare the tree-level S-matrix of $\mathcal{N}=4$ SYM theory to the scattering matrix of the two-dimensional field theories considered in Section 4.3. In particular, we are interested in two-to-two particle scattering processes. As indicated, the four-point amplitude of $\mathcal{N}=4$ SYM theory obeys

$$\widehat{\mathrm{J}}^a A_4 = f^a{}_{bc} \sum_{k=1}^{4} \sum_{j=1}^{k-1} \mathrm{J}_j^b \mathrm{J}_k^c \, A_4 = 0. \tag{6.41}$$

Here the generators of the Poincaré algebra enter the definition of the level-one generators $\widehat{\mathrm{J}}^a$, i.e. the Yangian symmetry is not merely a Yangian of an internal symmetry algebra (cf. (6.34)).

On the other hand, the two-particle S-matrix $\mathrm{S}(u)$ of the above two-dimensional theories is subject to (4.61), which for $u=0$ becomes

$$f^a{}_{bc} \mathrm{J}_1^b \mathrm{J}_2^c \, \mathrm{S}(0) + \mathrm{S}(0) \, f^a{}_{bc} \mathrm{J}_3^b \mathrm{J}_4^c = 0. \tag{6.42}$$

Here we think of the S-matrix as an operator that maps the ingcoming particles $3, 4$ to the outgoing particles $1, 2$. We set $u=0$ since the 2d rapidities are quantum numbers of the Poincaré algebra, which a priori does not form part of the 2d Yangian. In fact, one may rewrite (6.41) in the form (see [111] for more details)

$$\widehat{\mathrm{J}}^a A_4 = f^a{}_{bc} \mathrm{J}_1^b \mathrm{J}_2^c \, A_4 + f^a{}_{bc} \mathrm{J}_3^b \mathrm{J}_4^c \, A_4 = 0, \tag{6.43}$$

where it was used that the expression

$$f^a{}_{bc} \sum_{k=1}^{4} \sum_{j=1}^{k-1} \mathrm{J}_j^b \mathrm{J}_k^c - f^a{}_{bc} (\mathrm{J}_1^b \mathrm{J}_2^c + \mathrm{J}_3^b \mathrm{J}_4^c) = f^a{}_{bc} (\mathrm{J}_1^b \mathrm{J}_3^c + \mathrm{J}_1^b \mathrm{J}_4^c + \mathrm{J}_2^b \mathrm{J}_3^c + \mathrm{J}_2^b \mathrm{J}_4^c) \tag{6.44}$$

annihilates the four-point amplitude. Notably, (6.43) now looks very close to (6.42). This illustrates the similarity between the four-point amplitude A_4 of $\mathcal{N}=4$ SYM theory and the 2d

[70] The particular Yangian $Y[\mathfrak{psu}(2,2|4)]$ allows for further special features such as the occurence of so-called bonus or secret symmetries, see e.g. [107].

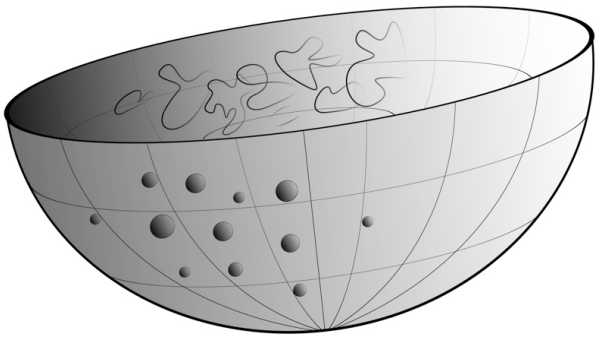


Figure 14: Illustration of the AdS/CFT correspondence: Gauge theory particles on the boundary of space are dual to gravity described by strings in the bulk. While the gauge theory lives in four-dimensional spacetime, the string theory is effectively described by a two-dimensional worldsheet theory.

S-matrix S(0) evaluated at 0. In fact, one may define a deformation $A_4(u)$ of $A_4(0) \equiv A_4$ [56,112], such that $A_4(u)$ transforms under an evaluation representation of the Yangian algebra with non-vanishing rapidity-parameter u in analogy to the 2d S-matrix $\mathrm{S}(u)$, cf. (4.61). While algebraically consistent, the physical interpretation of the 2d rapidity-like parameter u remains to be understood in the 4d theory.

6.4 Two Dimensions in Disguise: The AdS/CFT Correspondence

What makes $\mathcal{N} = 4$ SYM theory an outstanding example in the class of quantum gauge theories is its relation to gravity via string theory. The four-dimensional quantum field theory discussed in this section is conjectured to be the dual description of type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [113]. As such, it describes gravitational excitations via gauge degrees of freedom. The fact that the flat Minkowski background of the gauge theory represents the conformal boundary of the string geometry, makes this correspondence even more appealing, cf. Figure 14. The explicit map between the corresponding coupling parameters is given by

$$\lambda = g_{\mathrm{YM}}^2 N = \frac{R^4}{\alpha'^2}, \qquad\qquad \frac{1}{N} = \frac{4\pi g_{\mathrm{s}}}{\lambda}. \tag{6.45}$$

Here R denotes the common radius of AdS_5 and S^5 while α' represents the string tension. The string coupling constant is given by g_{s}.

In particular, the conjectured correspondence maps the strong coupling regime of the gauge theory to weak string coupling and vice versa. While explicit calculations in $\mathcal{N} = 4$ SYM theory require a small coupling expansion $\lambda \ll 1$, its string dual is only accessible for small curvature $R^4/\alpha'^2 = \lambda \gg 1$. This, on the one hand, represents an obstacle for proving the duality. On the other hand, weak coupling results in either gauge or string theory provide information on the strong coupling limit of its counterpart and thus open a new door to largely unexplored areas of research. This, however, requires a verification of the AdS/CFT correspondence which, for the moment, is most promising in the limit of large N, where the dual theories are believed to be integrable.

Notably, the super-string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is described by a two-dimensional worldsheet theory that vaguely resembles the principal chiral model briefly discussed in Section 2.4 (but is more complicated). Indeed, for this classical super-string theory nonlocal charges were constructed, which are similar to the ones of Section 2. This shows the classical integrability of

the string theory [114]. Also on the gauge theory side classical integrability has been discussed, formulating the equations of motion in the language of Lax pairs or the inverse scattering method [115] (see also [116]). However, $\mathcal{N} = 4$ SYM theory represents the first four-dimensional gauge theory that was found to be integrable and many of its features remain to be understood. Certainly, the Yangian plays an important role for this ongoing journey. In fact, Yangian symmetry was also observed for other AdS/CFT observables lying beyond the scope of this review, e.g. for 2d scattering matrices [117, 118], Wilson loops [119] or tree-level three-point functions [120].

7 Summary and Outlook

Hidden symmetries have great appeal. They explain mysterious simplifications and their identification poses exciting riddles. In these lectures we have discussed the Yangian, a particular class of hidden symmetry which appears in various physical contexts.

In Section 2, we first looked at classical field theories in two dimensions. We have seen that thinking outside the box of ordinary Noether symmetries, one may find nonlocal charges that were a priori hidden. After this classical discourse, the obvious question for the corresponding quantum symmetry arose. In order to understand this point, we made a step into a more mathematical direction. We followed Drinfel'd who defined the Yangian algebra to tackle an a priori unrelated problem, namely to solve the quantum Yang–Baxter equation. We have seen that addressing this problem leads to the rich mathematical framework of quantum groups and we have discussed the place of the Yangian in this context.

We then went back to 2d field theory in order to apply our supplemented mathematical background. With Lüscher, we understood how the quantum version of the above classical nonlocal symmetries can be defined by renormalizing their bilocal generators. A consistent definition of these charges at hand, we followed Bernard and identified them as the generators of the Yangian algebra and the field theory Lorentz boost as a realization of Drinfel'd's boost automorphism. We also realized that the scattering matrix of a 2d field theory with Yangian symmetry furnishes a solution to the quantum Yang–Baxter equation, i.e. the quantity that Drinfel'd was after when introducing the Yangian.

Having studied Yangian symmetry in the context of continuous two-dimensional field theories, we thought about a discretized version of the Yangian on spin chains. We understood the role played by local charges or Hamiltonians defining the spin chain dynamics and how these may co-exist with the nonlocal Yangian symmetry. Different boundary conditions were discussed and we indicated the existence of certain long-range spin chains and their connection to a generalization of Drinfel'd's boost automorphism.

Finally, we tried to better understand whether Yangian symmetry is tied to two dimensions. In Section 6 we briefly introduced the four-dimensional $\mathcal{N} = 4$ super Yang–Mills theory. We found that the color structure of this gauge theory in the planar limit allows to introduce a two-dimensional discrete space, namely the space of color traces, on which Yangian generators may be defined. As a consequence, we have seen the bulk Yangian symmetry of the theory's dilatation operator as well as the Yangian symmetry of tree-level scattering amplitudes. Lastly, we briefly sketched the duality of $\mathcal{N} = 4$ SYM theory to string theory described by a 2d worldsheet theory.

The Yangian provides in many respects a special and interesting realization of integrability. It represents one of three members within the family of integrable quantum group symmetries. In fact, the Yangian may be deformed and one obtains more general quantum groups, which

typically do not allow to scale away the quantum deformation parameter $\hbar$ (as it was possible for the Yangian discussed above). To be more precise, the solutions to the classical Yang–Baxter equation (and hence to its quantum deformation) fall into three categories via the Belavin–Drinfeld theorem [121] (cf. Appendix A): 1. Rational solutions, 2. Trigonometric solutions and 3. Elliptic solutions. Describing *rational* quantum R-matrices, the Yangian corresponds to the simplest of these categories and thereby to the most accessible mathematical structure. Hence, a lot remains to be discovered when going beyond this class.

Due to the limited scope of this review, certainly many interesting mathematical facts about the Yangian as well as physical applications of this algebra were not discussed or even touched in these lectures. For further reading on the Yangian and related topics, let us mention the very helpful and at many places complementary reviews by Bernard [33] and MacKay [78]. Also in the special context of the AdS/CFT correspondence, several useful reviews on Yangian symmetry exist, see e.g. [75, 106, 118, 122]. Note also the more general collection of reviews on integrability in AdS/CFT [123].

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A Belavin–Drinfeld Theorem

According to a theorem by Belavin and Drinfel'd, the (nondengenerate) solutions $r(u)$ to the classical Yang–Baxter equation can be classified via the discrete subgroup $\Gamma \subset \mathbb{C}$ of their poles in the complex plane [121] (see also [53]). One finds three different categories:

1. Rational functions with $\mathrm{rank}(\Gamma) = 0$.
2. Trigonormetric functions with $\mathrm{rank}(\Gamma) = 1$, i.e. functions of the form $f(e^{ku})$ with f being a rational function.
3. Elliptic functions with $\mathrm{rank}(\Gamma) = 2$.

The Yangian corresponds to quantum deformations of the first class of solutions. The last category leads to elliptic quantum groups while the second class corresponds to quantum affine algebras related to trigonometric R-matrices.

References

[1] R. P. Feynman, *"Difficulties in Applying the Variational Principle to Quantum Field Theories"*, in Variational Calculations in Quantum Field Theory Chapter 3, 28. • M. T. Batchelor, *"The Bethe ansatz after 75 years"*, Physics today 60, 36 (2007).

[2] B. Sutherland, *"Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems"*, World Scientific Press (2004). • J.-S. Caux and J. Mossel, *"Remarks on the notion of quantum integrability"*, J. Stat. Mech. 1102, P02023 (2011), arxiv:1012.3587, in: *"Proceedings, 24th IUPAP International Conference on Statistical Physics (STATPHYS24)"*, pp. P02023.

[3] V. Drinfeld, *"Hopf algebras and the quantum Yang-Baxter equation"*, Sov. Math. Dokl. 32, 254 (1985).

[4] V. Drinfeld, *"Quantum groups"*, Proceedings of the International Congress of Mathematics Berkeley, (1986).

[5] V. G. Drinfeld, *"Quantum groups"*, J. Sov. Math. 41, 898 (1988), [Zap. Nauchn. Semin.155,18(1986)].

[6] V. Drinfeld, *"A New realization of Yangians and quantized affine algebras"*, Sov. Math. Dokl. 36, 212 (1988).

[7] M. Jimbo, *"A q difference analog of U(g) and the Yang-Baxter equation"*, Lett. Math. Phys. 10, 63 (1985).

[8] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *"Quantization of Lie Groups and Lie Algebras"*, Leningrad Math. J. 1, 193 (1990), [Alg. Anal.1,178(1989)].

[9] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *"Quantum Inverse Scattering Method and Correlation Functions"*.

[10] S. R. Coleman and J. Mandula, *"All Possible Symmetries of the S Matrix"*, Phys. Rev. 159, 1251 (1967).

[11] D. Bernard, *"Hidden Yangians in 2-D massive current algebras"*, Commun. Math. Phys. 137, 191 (1991).

[12] A. LeClair and F. Smirnov, *"Infinite quantum group symmetry of fields in massive 2-D quantum field theory"*, Int. J. Mod. Phys. A7, 2997 (1992), hep-th/9108007.

[13] D. J. Gross and A. Neveu, *"Dynamical Symmetry Breaking in Asymptotically Free Field Theories"*, Phys. Rev. D10, 3235 (1974).

[14] M. Tetelman, *"Lorentz group for two-dimensional integrable lattice systems"*, Sov. Phys. JETP. 55, 306 (1982).

[15] H. B. Thacker, *"Corner Transfer Matrices and Lorentz Invariance on a Lattice"*, Physica 18D, 348 (1986).

[16] T. Bargheer, N. Beisert and F. Loebbert, *"Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains"*, J. Stat. Mech. 0811, L11001 (2008), arxiv:0807.5081. • T. Bargheer, N. Beisert and F. Loebbert, *"Long-Range Deformations for Integrable Spin Chains"*, J. Phys. A42, 285205 (2009), arxiv:0902.0956.

[17] J. A. Minahan and K. Zarembo, *"The Bethe-ansatz for N = 4 super Yang-Mills"*, JHEP 0303, 013 (2003), hep-th/0212208.

[18] N. Beisert, C. Kristjansen and M. Staudacher, *"The dilatation operator of N = 4 super Yang-Mills theory"*, Nucl. Phys. B664, 131 (2003), hep-th/0303060.

[19] N. Beisert and M. Staudacher, *"The N=4 SYM Integrable Super Spin Chain"*, Nucl. Phys. B670, 439 (2003), hep-th/0307042.

[20] L. Dolan, C. R. Nappi and E. Witten, *"A Relation Between Approaches to Integrability in Superconformal Yang-Mills Theory"*, JHEP 0310, 017 (2003), hep-th/0308089.

[21] L. Dolan, C. R. Nappi and E. Witten, *"Yangian symmetry in D = 4 superconformal Yang-Mills theory"*, hep-th/0401243.

[22] J. M. Drummond, J. M. Henn and J. Plefka, *"Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory"*, JHEP 0905, 046 (2009), arxiv:0902.2987, in: *"Strangeness in quark matter. Proceedings, International Conference, SQM 2008, Beijing, P.R. China, October 5-10, 2008"*, pp. 046.

[23] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert and T. McLoughlin, *"Exacting $\mathcal{N} = 4$ Superconformal Symmetry"*, JHEP 0911, 056 (2009), arxiv:0905.3738.

[24] N. Beisert, J. Henn, T. McLoughlin and J. Plefka, *"One-Loop Superconformal and Yangian Symmetries of Scattering Amplitudes in N=4 Super Yang-Mills"*, JHEP 1004, 085 (2010), arxiv:1002.1733.

[25] S. Caron-Huot and S. He, *"Jumpstarting the All-Loop S-Matrix of Planar N=4 Super Yang-Mills"*, JHEP 1207, 174 (2012), arxiv:1112.1060.

[26] A. Torrielli, *"Lectures on Classical Integrability"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[27] F. Loebbert, *"Lectures on Yangian Symmetry"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[28] D. Bombardelli, *"Lectures on S-Matrices and Integrability"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[29] F. Levkovich-Maslyuk, *"Lectures on the Bethe Ansatz"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[30] S. J. van Tongeren, *"Introduction to the Thermodynamic Bethe Ansatz"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear. • S. Negro, *"Lectures on Integrable Structures in Quantum Field Theory and Massive ODE/IM Correspondence"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[31] D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, A. Sfondrini, S. Negro, I. M. Szécsényi, S. J. van Tongeren and A. Torrielli, *"An Integrability Primer for the Gauge-Gravity Correspondence: an Introduction"*, arxiv:1605.xxxxx, in: *"An Integrability Primer for the Gauge-Gravity Correspondence"*, ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[32] N. J. MacKay, *"On the classical origins of Yangian symmetry in integrable field theory"*, Phys. Lett. B281, 90 (1992), [Erratum: Phys. Lett.B308,444(1993)].

[33] D. Bernard, *"An Introduction to Yangian Symmetries"*, Int. J. Mod. Phys. B7, 3517 (1993), hep-th/9211133.

[34] E. Brezin, C. Itzykson, J. Zinn-Justin and J. Zuber, *"Remarks About the Existence of Nonlocal Charges in Two-Dimensional Models"*, Phys. Lett. B82, 442 (1979).

[35] M. Luscher and K. Pohlmeyer, *"Scattering of Massless Lumps and Nonlocal Charges in the Two-Dimensional Classical Nonlinear Sigma Model"*, Nucl. Phys. B137, 46 (1978).

[36] M. Luscher, *"Quantum Nonlocal Charges and Absence of Particle Production in the Two-Dimensional Nonlinear Sigma Model"*, Nucl. Phys. B135, 1 (1978).

[37] O. Babelon, D. Bernard and M. Talon, *"Introduction to Classical Integrable Systems"*, Cambridge University Press (2003).

[38] H. de Vega, *"Field Theories With an Infinite Number of Conservation Laws and Backlund Transformations in Two-dimensions"*, Phys. Lett. B87, 233 (1979). • H. Eichenherr and M. Forger, *"On the Dual Symmetry of the Nonlinear Sigma Models"*, Nucl. Phys. B155, 381 (1979). • V. Zakharov and A. Mikhailov, *"Example of nontrivial soliton interaction in two-dimensional classical field theory"*, Pisma Zh. Eksp. Teor. Fiz. 27, 47 (1978).

[39] H. de Vega, H. Eichenherr and J. Maillet, *"Canonical Charge Algebras for Integrable Fermionic Theories"*, Phys. Lett. B132, 337 (1983).

[40] Y. Nambu and G. Jona-Lasinio, *"Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. 1."*, Phys. Rev. 122, 345 (1961). • Y. Nambu and G. Jona-Lasinio, *"Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. II"*, Phys. Rev. 124, 246 (1961).

[41] S. R. Coleman, *"There are no Goldstone bosons in two-dimensions"*, Commun. Math. Phys. 31, 259 (1973).

[42] H. de Vega, H. Eichenherr and J. Maillet, *"Yang-Baxter Algebras of Monodromy Matrices in Integrable Quantum Field Theories"*, Nucl. Phys. B240, 377 (1984).

[43] T. L. Curtright and C. K. Zachos, *"Nonlocal Symmetry Ward Identities"*, Phys. Rev. D24, 2661 (1981).

[44] L. Dolan and A. Roos, *"Nonlocal Currents as Noether Currents"*, Phys. Rev. D22, 2018 (1980).

[45] B.-y. Hou, M.-l. Ge and Y.-s. Wu, *"Noether Analysis for the Hidden Symmetry Responsible for Infinite Set of Nonlocal Currents"*, Phys. Rev. D24, 2238 (1981).

[46] A. M. Polyakov, *"Gauge Fields as Rings of Glue"*, Nucl. Phys. B164, 171 (1980).

[47] C.-N. Yang, *"Some exact results for the many body problems in one dimension with repulsive delta function interaction"*, Phys. Rev. Lett. 19, 1312 (1967).

[48] R. J. Baxter, *"Partition function of the eight vertex lattice model"*, Annals Phys. 70, 193 (1972).

[49] J. B. McGuire, *"Study of Exactly Soluble One-Dimensional N-Body Problems"*, Journal of Mathematical Physics 5, 622 (1964).

[50] M. Gaudin, *"Un systeme a une dimension de fermions en interaction"*, Physics Letters A 24, 55 (1967). • B. Sutherland, *"Further Results for the Many-Body Problem in One Dimension"*, Phys. Rev. Lett. 20, 98 (1968). • N. Oelkers, M. T. Batchelor, M. Bortz and X.-W. Guan, *"Bethe ansatz study of one-dimensional Bose and Fermi gases with periodic and hard wall boundary conditions"*, Journal of Physics A Mathematical General 39, 1073 (2006), `cond-mat/0511694`. • X.-W. Guan, M. T. Batchelor and C. Lee, *"Fermi gases in one dimension: From Bethe ansatz to experiments"*, Reviews of Modern Physics 85, 1633 (2013), `arxiv:1301.6446`.

[51] H. Bethe, *"On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain"*, Z. Phys. 71, 205 (1931).

[52] E. H. Lieb and W. Liniger, *"Exact analysis of an interacting Bose gas. 1. The General solution and the ground state"*, Phys. Rev. 130, 1605 (1963).

[53] M. Jimbo, *"Introduction to the Yang-Baxter Equation"*, Int. J. Mod. Phys. A4, 3759 (1989).

[54] V. Chari and A. N. Pressley, *"A Guide to Quantum Groups"*, Cambridge University Press (1995).

[55] T. Bargheer, F. Loebbert and C. Meneghelli, *"Symmetries of Tree-level Scattering Amplitudes in N=6 Superconformal Chern-Simons Theory"*, Phys. Rev. D82, 045016 (2010), `arxiv:1003.6120`.

[56] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka and M. Staudacher, *"Harmonic R-matrices for Scattering Amplitudes and Spectral Regularization"*, Phys. Rev. Lett. 110, 121602 (2013), `arxiv:1212.0850`.

[57] T. Bargheer, Y.-t. Huang, F. Loebbert and M. Yamazaki, *"Integrable Amplitude Deformations for N=4 Super Yang-Mills and ABJM Theory"*, Phys. Rev. D91, 026004 (2015), arxiv:1407.4449.

[58] J. A. Fuchs, *"Affine Lie Algebras and Quantum Groups"*, Cambridge University Press (1995).

[59] P. Aschieri, *"Lectures on Hopf Algebras, Quantum Groups and Twists"*, hep-th/0703013, in: *"2nd Modave Summer School in Theoretical Physics Modave, Belgium, August 6-12, 2006"*.

[60] S. Z. Levendorskiĭ, *"On generators and defining relations of Yangians"*, Journal of Geometry and Physics 12, 1 (1993).

[61] L. Takhtajan and L. Faddeev, *"The Quantum method of the inverse problem and the Heisenberg XYZ model"*, Russ. Math. Surveys 34, 11 (1979).

[62] A. Molev, *"Yangians and classical Lie algebras"*, Mathematical Surveys and Monographs 143, A. Molev (2007).

[63] L. Dolan and C. R. Nappi, *"Spin models and superconformal Yang-Mills theory"*, Nucl. Phys. B717, 361 (2005), hep-th/0411020.

[64] P. Kulish and E. Sklyanin, *"Quantum Spectral Transform Method. Recent Developments."*, Lect. Notes Phys. 151, 61 (1982).

[65] T. Hauer, *"Massive current algebra in the many flavor chiral Gross-Neveu model"*, Nucl. Phys. B502, 436 (1997), hep-th/9702016.

[66] E. Abdalla, M. C. B. Abdalla and K. D. Rothe, *"Nonperturbative methods in two-dimensional quantum field theory"*, World Scientific (1991).

[67] A. Belavin, *"A Direct calculation of the spectrum of masses in an integrable model from the Hopf algebra symmetry"*, Phys. Lett. B283, 67 (1992).

[68] N. J. MacKay, *"On the bootstrap structure of Yangian invariant factorized S matrices"*, hep-th/9211091.

[69] G. W. Delius and N. J. MacKay, *"Quantum group symmetry in sine-Gordon and affine Toda field theories on the half line"*, Commun. Math. Phys. 233, 173 (2003), hep-th/0112023.

[70] P. Dorey, *"Exact S matrices"*, hep-th/9810026, in: *"Conformal field theories and integrable models. Proceedings, Eotvos Graduate Course, Budapest, Hungary, August 13-18, 1996"*, pp. 85-125.

[71] E. K. Sklyanin, *"Quantum inverse scattering method. Selected topics"*, hep-th/9211111.

[72] L. D. Faddeev, *"How Algebraic Bethe Ansatz works for integrable model"*, hep-th/9605187.

[73] N. Beisert and D. Erkal, *"Yangian symmetry of long-range gl(N) integrable spin chains"*, J. Stat. Mech. 0803, P03001 (2008), arxiv:0711.4813.

[74] F. Haldane, Z. Ha, J. Talstra, D. Bernard and V. Pasquier, *"Yangian symmetry of integrable quantum chains with long range interactions and a new description of states in conformal field theory"*, Phys. Rev. Lett. 69, 2021 (1992).

[75] N. Beisert, *"On Yangian Symmetry in Planar N=4 SYM"*, arxiv:1004.5423, in: *"Quantum chromodynamics and beyond: Gribov-80 memorial volume. Proceedings, Memorial Workshop devoted to the 80th birthday of V.N. Gribov, Trieste, Italy, May 26-28, 2010"*, http://inspirehep.net/record/853644/files/arXiv:1004.5423.pdf.

[76] R. Frassek, *"Q-operators, Yangian invariance and the quantum inverse scattering method"*, arxiv:1412.3339, http://inspirehep.net/record/1333697/files/arXiv:1412.3339.pdf.

[77] G. Delius, N. MacKay and B. Short, *"Boundary remnant of Yangian symmetry and the structure of rational reflection matrices"*, Phys. Lett. B522, 335 (2001), hep-th/0109115. • N. MacKay and C. Young, *"Classically integrable boundary conditions for symmetric space sigma models"*,

Phys. Lett. B588, 221 (2004), hep-th/0402182. • N. MacKay and V. Regelskis, *"Yangian symmetry of the Y=0 maximal giant graviton"*, JHEP 1012, 076 (2010), arxiv:1010.3761.

[78] N. MacKay, *"Introduction to Yangian symmetry in integrable field theory"*, Int. J. Mod. Phys. A20, 7189 (2005), hep-th/0409183.

[79] R. I. Nepomechie, *"A Spin chain primer"*, Int. J. Mod. Phys. B13, 2973 (1999), hep-th/9810032, in: *"New symmetries in statistical mechanics and condensed matter physics. Proceedings, Euroconference, Torino, Italy, July 20-August 1, 1998"*, pp. 2973-2986.

[80] V. O. Tarasov, L. A. Takhtajan and L. D. Faddeev, *"Local Hamiltonians for integrable quantum models on a lattice"*, Theor. Math. Phys. 57, 1059 (1983), [Teor. Mat. Fiz.57,163(1983)].

[81] R. J. Baxter, *"One-dimensional anisotropic Heisenberg chain"*, Annals Phys. 70, 323 (1972).

[82] M. Luscher, *"Dynamical Charges in the Quantized, Renormalized Massive Thirring Model"*, Nucl. Phys. B117, 475 (1976).

[83] A. S. Fokas and B. Fuchssteiner, *"The Hierarchy of the Benjamin-Ono equation"*, Phys. Lett. A86, 341 (1981). • B. Fuchssteiner, *"Master symmetries, higher order time-dependent symmetries and conserved densities of non-linear evolution equations"*, Progr. Theor. Phys. 70, 1508 (1983).

[84] J. M. Evans, M. Hassan, N. J. MacKay and A. J. Mountain, *"Local conserved charges in principal chiral models"*, Nucl. Phys. B561, 385 (1999), hep-th/9902008.

[85] N. Beisert, L. Fiévet, M. de Leeuw and F. Loebbert, *"Integrable Deformations of the XXZ Spin Chain"*, J. Stat. Mech. 2013, P09028 (2013), arxiv:1308.1584.

[86] F. Loebbert, *"Recursion Relations for Long-Range Integrable Spin Chains with Open Boundary Conditions"*, Phys. Rev. D85, 086008 (2012), arxiv:1201.0888.

[87] H. Itoyama and H. B. Thacker, *"Lattice Virasoro Algebra and Corner Transfer Matrices in the Baxter Eight Vertex Model"*, Phys. Rev. Lett. 58, 1395 (1987). • Y. Jiang, I. Kostov, F. Loebbert and D. Serban, *"Fixing the Quantum Three-Point Function"*, JHEP 1404, 019 (2014), arxiv:1401.0384.

[88] D. Serban and M. Staudacher, *"Planar N=4 gauge theory and the Inozemtsev long range spin chain"*, JHEP 0406, 001 (2004), hep-th/0401057.

[89] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *"Large N field theories, string theory and gravity"*, Phys. Rept. 323, 183 (2000), hep-th/9905111. • G. 't Hooft, *"Large N"*, hep-th/0204069, in: *"In *Tempe 2002, Phenomenology of large N(c) QCD* 3-18"*, pp. 3-18. • N. Beisert, *"The Dilatation operator of N=4 super Yang-Mills theory and integrability"*, Phys. Rept. 405, 1 (2004), hep-th/0407277. • J. A. Minahan, *"A brief introduction to the Bethe ansatz in N=4 super-Yang-Mills"*, J. Phys. A39, 12657 (2006).

[90] F. Gliozzi, J. Scherk and D. I. Olive, *"Supersymmetry, Supergravity Theories and the Dual Spinor Model"*, Nucl. Phys. B122, 253 (1977).

[91] L. Brink, J. H. Schwarz and J. Scherk, *"Supersymmetric Yang-Mills Theories"*, Nucl. Phys. B121, 77 (1977).

[92] N. Beisert, *"Review of AdS/CFT Integrability, Chapter VI.1: Superconformal Symmetry"*, Lett. Math. Phys. 99, 529 (2012), arxiv:1012.4004.

[93] P. Fayet, *"Fermi-Bose Hypersymmetry"*, Nucl. Phys. B113, 135 (1976).

[94] M. F. Sohnius and P. C. West, *"Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory"*, Phys. Lett. B100, 245 (1981).

[95] S. Mandelstam, *"Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model"*, Nucl. Phys. B213, 149 (1983).

[96] P. S. Howe, K. S. Stelle and P. K. Townsend, *"Miraculous Ultraviolet Cancellations in Supersymmetry Made Manifest"*, Nucl. Phys. B236, 125 (1984). • L. Brink, O. Lindgren and B. E. W. Nilsson, *"N=4 Yang-Mills Theory on the Light Cone"*, Nucl. Phys. B212, 401 (1983).

[97] G. 't Hooft, *"A Planar Diagram Theory for Strong Interactions"*, Nucl. Phys. B72, 461 (1974).

[98] S. Fubini, A. J. Hanson and R. Jackiw, *"New approach to field theory"*, Phys. Rev. D7, 1732 (1973). • S. Minwalla, *"Restrictions imposed by superconformal invariance on quantum field theories"*, Adv. Theor. Math. Phys. 2, 781 (1998), `hep-th/9712074`.

[99] N. Beisert, *"The complete one loop dilatation operator of N=4 superYang-Mills theory"*, Nucl. Phys. B676, 3 (2004), `hep-th/0307015`.

[100] B. I. Zwiebel, *"Yangian Symmetry at Two-Loops for the su(2|1) Sector of N=4 SYM"*, J. Phys. A40, 1141 (2007), `hep-th/0610283`.

[101] V. P. Nair, *"A current algebra for some gauge theory amplitudes"*, Phys. Lett. B214, 215 (1988).

[102] L. Brink, O. Lindgren and B. E. W. Nilsson, *"$\mathcal{N} = 4$ Yang-Mills theory on the light cone"*, Nucl. Phys. B212, 401 (1983).

[103] M. T. Grisaru, H. N. Pendleton and P. van Nieuwenhuizen, *"Supergravity and the S Matrix"*, Phys. Rev. D15, 996 (1977). • M. T. Grisaru and H. N. Pendleton, *"Some Properties of Scattering Amplitudes in Supersymmetric Theories"*, Nucl. Phys. B124, 81 (1977).

[104] S. J. Parke and T. R. Taylor, *"An Amplitude for n Gluon Scattering"*, Phys. Rev. Lett. 56, 2459 (1986). • F. A. Berends and W. T. Giele, *"Recursive Calculations for Processes with n Gluons"*, Nucl. Phys. B306, 759 (1988).

[105] E. Witten, *"Perturbative gauge theory as a string theory in twistor space"*, Commun. Math. Phys. 252, 189 (2004), `hep-th/0312171`.

[106] T. Bargheer, N. Beisert and F. Loebbert, *"Exact Superconformal and Yangian Symmetry of Scattering Amplitudes"*, J. Phys. A44, 454012 (2011), `arxiv:1104.0700`.

[107] T. Matsumoto, S. Moriyama and A. Torrielli, *"A Secret Symmetry of the AdS/CFT S-matrix"*, JHEP 0709, 099 (2007), `arxiv:0708.1285`. • N. Beisert and B. U. W. Schwab, *"Bonus Yangian Symmetry for the Planar S-Matrix of N=4 Super Yang-Mills"*, Phys. Rev. Lett. 106, 231602 (2011), `arxiv:1103.0646`. • M. de Leeuw, T. Matsumoto, S. Moriyama, V. Regelskis and A. Torrielli, *"Secret Symmetries in AdS/CFT"*, Phys. Scripta 02, 028502 (2012), `arxiv:1204.2366`.

[108] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *"Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory"*, Nucl. Phys. B828, 317 (2010), `arxiv:0807.1095`.

[109] B. I. Zwiebel, *"From Scattering Amplitudes to the Dilatation Generator in N=4 SYM"*, J. Phys. A45, 115401 (2012), `arxiv:1111.0083`.

[110] M. Wilhelm, *"Amplitudes, Form Factors and the Dilatation Operator in $\mathcal{N} = 4$ SYM Theory"*, JHEP 1502, 149 (2015), `arxiv:1410.6309`.

[111] A. Brandhuber, P. Heslop, G. Travaglini and D. Young, *"Yangian Symmetry of Scattering Amplitudes and the Dilatation Operator in $N = 4$ Supersymmetric Yang-Mills Theory"*, Phys. Rev. Lett. 115, 141602 (2015), `arxiv:1507.01504`.

[112] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka and M. Staudacher, *"Spectral Parameters for Scattering Amplitudes in N=4 Super Yang-Mills Theory"*, JHEP 1401, 094 (2014), `arxiv:1308.3494`.

[113] J. M. Maldacena, *"The large N limit of superconformal field theories and supergravity"*, Adv. Theor. Math. Phys. 2, 231 (1998), `hep-th/9711200`.

[114] I. Bena, J. Polchinski and R. Roiban, *"Hidden symmetries of the AdS(5) x S**5 superstring"*, Phys. Rev. D69, 046002 (2004), hep-th/0305116. • M. Hatsuda and K. Yoshida, *"Classical integrability and super Yangian of superstring on AdS(5) x S**5"*, Adv. Theor. Math. Phys. 9, 703 (2005), hep-th/0407044.

[115] I. V. Volovich, *"Supersymmetric Yang–Mills Equations as an Inverse Scattering Problem"*, Lett. Math. Phys. 7, 517 (1983). • I. V. Volovich, *"Supersymmetric Yang–Mills Theory and the Inverse Scattering Method"*, Theor. Math. Phys. 57, 1269 (1983), [Teor. Mat. Fiz.57,469(1983)]. • I. Ya. Arefeva and I. V. Volovich, *"Higher Conservation Laws for Supersymmetric Gauge Theories"*, Phys. Lett. B149, 131 (1984). • E. Abdalla, M. Forger and M. Jacques, *"Higher Conservation Laws for Ten-dimensional Supersymmetric Yang-Mills Theories"*, Nucl. Phys. B307, 198 (1988), in: *"In Clausthal 1986, Proceedings, Differential Geometric Methos in Theoretical Physics 162., In Marseille 1986, Proceedings, Mathematical Physics 641-643"*, pp. 198.

[116] E. Witten, *"An Interpretation of Classical Yang-Mills Theory"*, Phys. Lett. B77, 394 (1978).

[117] N. Beisert, *"The S-matrix of AdS / CFT and Yangian symmetry"*, PoS SOLVAY, 002 (2006), arxiv:0704.0400, in: *"Proceedings, Bethe ansatz: 75 years later"*, pp. 002. • M. de Leeuw, *"Bound States, Yangian Symmetry and Classical r-matrix for the AdS(5) x S**5 Superstring"*, JHEP 0806, 085 (2008), arxiv:0804.1047. • G. Arutyunov, M. de Leeuw and A. Torrielli, *"The Bound State S-Matrix for AdS(5) x S**5 Superstring"*, Nucl. Phys. B819, 319 (2009), arxiv:0902.0183.

[118] A. Torrielli, *"Yangians, S-matrices and AdS/CFT"*, J. Phys. A44, 263001 (2011), arxiv:1104.2474.

[119] D. Müller, H. Münkler, J. Plefka, J. Pollok and K. Zarembo, *"Yangian Symmetry of smooth Wilson Loops in $\mathcal{N} = 4$ super Yang-Mills Theory"*, JHEP 1311, 081 (2013), arxiv:1309.1676. • H. Münkler and J. Pollok, *"Minimal surfaces of the $AdS_5 \times S^5$ superstring and the symmetries of super Wilson loops at strong coupling"*, J. Phys. A48, 365402 (2015), arxiv:1503.07553. • H. Münkler, *"Bonus Symmetry for Super Wilson Loops"*, arxiv:1507.02474. • N. Beisert, D. Muller, J. Plefka and C. Vergu, *"Integrability of Smooth Wilson Loops in N=4 Superspace"*, arxiv:1509.05403.

[120] Y. Kazama, S. Komatsu and T. Nishimura, *"Novel construction and the monodromy relation for three-point functions at weak coupling"*, JHEP 1501, 095 (2015), arxiv:1410.8533, [Erratum: JHEP08,145(2015)]. • Y. Jiang, I. Kostov, A. Petrovskii and D. Serban, *"String Bits and the Spin Vertex"*, Nucl. Phys. B897, 374 (2015), arxiv:1410.8860.

[121] A. A. Belavin and V. G. Drinfel'd, *"Solutions of the Classical Yang–Baxter Equation for Simple Lie Algebras"*, Funct. Anal. Appl. 159, V. G. Drinfel'd (1982).

[122] F. Spill, *"Yangians in Integrable Field Theories, Spin Chains and Gauge-String Dualities"*, Rev. Math. Phys. 24, 1230001 (2012), arxiv:1201.1884.

[123] N. Beisert et al., *"Review of AdS/CFT Integrability: An Overview"*, Lett. Math. Phys. 99, 3 (2012), arxiv:1012.3982.

Lectures on S-matrices and Integrability

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Abstract

In these notes we review the S-matrix theory in (1+1)-dimensional integrable models, focusing mainly on the relativistic case. Once the main definitions and physical properties are introduced, we discuss the factorization of scattering processes due to integrability. We then focus on the analytic properties of the 2-particle scattering amplitude and illustrate the derivation of the S-matrices for all the possible bound states using the so-called bootstrap principle. General algebraic structures underlying the S-matrix theory and its relation with the form factors axioms are briefly mentioned. Finally, we discuss the S-matrices of sine-Gordon and $SU(2)$, $SU(3)$ chiral Gross-Neveu models.

This is part of a collection of lecture notes for the *Young Researchers Integrability School*, organised by the GATIS network at Durham University on 6-10 July 2015.

In loving memory of Lilia Grandi

Contents

1 Introduction

The *S-matrix* program is a non-perturbative analytic approach to the scattering problem in quantum field theory (QFT), whose origins date back to the works by Wheeler [1] and Heisenberg [2]. The main purpose of the program was to overcome the problems of QFT related on one hand to the divergences emerging from standard perturbative methods, on the other hand to the discovery, in the '50s and '60s, of many hadronic resonances with high spin.

The idea, further developed by Chew [3], Mandelstam [4] and many others, was to compute scattering amplitudes and mass spectra without the use of a Lagrangian formulation, by imposing analytic constraints on the S-matrix, that is the operator relating initial and final states in a scattering process, and by giving a physical interpretation of all its singularities. Moreover, higher spin particles were treated on the same footing as the fundamental ones. This latter aspect will be illustrated in these notes when the so-called *bootstrap principle* will be discussed.

Unfortunately, after the initial successes, not many quantitative results were obtained in real-world particle physics. Moreover, the search for exact S-matrix models was finally discouraged by the Coleman-Mandula theorem [5], stating that QFT models in $d > 2$ space-time dimensions, with higher-order conserved charges, can only have trivial S-matrices.

However, between the '70s and the '80s the program was given a new boost in the context of $d = 2$ *integrable* theories, whose S-matrices are non trivial and can be uniquely fixed. Furthermore, as we will see, knowing the amplitudes for the scattering between 2-particle states is sufficient, at least in principle, to reconstruct the correlation functions of the theory, through the *form factor* program.

The S-matrix plays an essential rôle also in calculating the spectrum of integrable theories. Both in the large volume approximation, through the derivation of the asymptotic Bethe ansatz, reviewed in [6], and at finite volume, being the key ingredient of techniques like the Lüscher formulas [7] and the thermodynamic Bethe ansatz (TBA), reviewed in [8].

We believe that this is one of the strongest reasons to study the integrable S-matrix theory, that is the subject of these lectures. In particular, we will try to describe in a pedagogical way some of the fundamental concepts developed in this research field, assuming that the reader is familiar with the basics of quantum mechanics and special relativity. In order to give a deeper understanding of a few technical aspects, calculations will be described in full detail in some simple cases only. These tools and ideas can be then adapted to the study of much more complicated systems and the interested reader may find more complete and advanced discussions on the many important applications of such techniques in the quoted references. Since we could not cover the enormous literature on the subject, we selected a few reviews and original papers concerning the models discussed in these notes.

In particular, the lectures focused mostly on *relativistic* cases and were built mainly on the book by Mussardo [9], the lectures by Dorey [10] and the paper by Zamolodchikov and Zamolodchikov [11]. For the important non-relativistic case of AdS_5/CFT_4, the reading of [12], especially Chapter 3, is suggested, as well as the reviews [13, 14] and the seminal papers [15–19], and [20] for an overview of many other recent developments in the context of gauge/gravity dualities.

The *outline* of the lectures is the following: after the introduction of the necessary definitions, with a brief description of the S-matrix physical properties, the main ideas underlying the demonstration of the factorization property for integrable S-matrices will be explained.

Then we will focus on 2-particle S-matrices, including those for the processes involving bound states, and on their analytic and algebraic properties.

A few examples, regarding 2-particle S-matrices of the sine-Gordon and chiral Gross-Neveu models, will be given.

The latter theories will be used also to explain the links between S-matrices and correlation functions, through a very introductory discussion on the *form factor* program in integrable models. This part is built mainly on the paper [21] and the review [22] (see also [23], [24] and the book [25]).

Finally, we conclude with a short guide to the literature about recent developments on S-matrices in AdS/CFT correspondences.

2 Asymptotic states and S-matrix

2.1 Definitions

It is well known that in quantum mechanics the time evolution of a system can be defined through an unitary operator $U(t, t_0)$, which generates the state $|\psi(t)\rangle$ by acting on a state $|\psi(t_0)\rangle$:

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \,. \tag{2.1}$$

In order to study a scattering process, actually, it is not necessary to know $U(t, t_0)$ at any values of t, t_0, but it is enough to know it at $t_0 \to -\infty$ and $t \to +\infty$. Indeed, if we assume that interactions among particles occur in a very small region of the space-time, then, very far from the interaction region, we can treat them as free particles. Thus we need to define in a formal way these quantum states of free excitations introducing the so-called *asymptotic states*

$$|p_1, p_2, \ldots, p_n\rangle^{in/out}_{a_1 a_2 \ldots a_n} \,, \tag{2.2}$$

where n is the number of particles, p_i are their momenta and indices a_i label their flavors. Essentially, the asymptotic states describe wave packets with approximate positions at given times: in particular, n free particles at time $t \to -\infty$ for the *in* states and at $t \to +\infty$ for the *out* ones. We choose the order of momenta to be $p_1 > p_2 > \cdots > p_n$. Any intermediate state can equivalently be expanded on the *in* or *out* bases.
The S-matrix is defined as the linear operator that maps final asymptotic states into initial asymptotic states (or vice versa, depending on the convention adopted, related to the inversion of such operator):

$$|\ldots\rangle_{in} = S|\ldots\rangle_{out} \,. \tag{2.3}$$

Written in components, this reads

$$\begin{aligned} &|p_1, p_2, \ldots, p_n\rangle^{in}_{a_1, a_2, \ldots, a_n} \\ &= \sum_{m=2}^{\infty} \sum_{\substack{p'_1 > \cdots > p'_m \\ b_1, \ldots, b_m}} S^{b_1, \ldots, b_m}_{a_1, \ldots, a_n}(p_1, \ldots, p_n; p'_1, \ldots, p'_m) \, |p'_1, p'_2 \ldots, p'_m\rangle^{out}_{b_1, b_2, \ldots, b_m} \,, \end{aligned} \tag{2.4}$$

where the second line actually involves integrals in $p'_1, p'_2 \ldots, p'_m$.
Hence S is the time evolution operator from $t = -\infty$ to $t = +\infty$:

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} U(t, t_0) \,. \tag{2.5}$$

If the system has an Hamiltonian

$$H = H_0 + H_I \,, \tag{2.6}$$

where H_0 is the Hamiltonian of the free system and $H_I = H_I(t)$ is the interaction part in interaction (Dirac) picture[1], then S can be expressed as

$$S = \mathcal{T} \exp\left[-\mathrm{i}\int_{-\infty}^{+\infty} dt H_I(t)\right], \tag{2.7}$$

where $\mathcal{T}$ denotes the time-ordering for the series expansion of the exponential in (2.7).

2.2 General properties

In this Section we discuss some general assumptions motivated by physical properties fulfilled by usual QFTs. As previously mentioned, *interactions* among particles are assumed to occur only at *short range*. Another obvious assumption is the validity of the *QM superposition principle*, meaning that asymptotic states form a complete basis for initial and final states and any *in* state can be expanded in the basis of *out* states and vice versa, through the time evolution linear operator S, as expressed by (2.3). Moreover, *probability conservation* implies that

$$1 = \sum_m |\langle m|S|\psi\rangle|^2 , \tag{2.8}$$

where $|\psi\rangle = \sum_n a_n |n\rangle$ and $|m\rangle$, $|n\rangle$ are orthogonal, complete basis vectors generating the Hilbert space of the asymptotic states. Then one can show that

$$1 = \sum_m |\langle m|S|\psi\rangle|^2 = \sum_m \langle\psi|S^\dagger|m\rangle\langle m|S|\psi\rangle = \langle\psi|S^\dagger S|\psi\rangle = \sum_{n,m} a_n^* a_m \langle n|S^\dagger S|m\rangle , \tag{2.9}$$

meaning that the S-matrix has to be *unitary*: $S^\dagger S = 1$. We will refer to this property also as *physical unitarity*. Working mainly with relativistic theories, we will be interested in the consequences of *Lorentz invariance*. In particular, given a generic Lorentz transformation denoted by $L|m\rangle = |m'\rangle$, requiring invariance under such transformation at the level of the S-matrix is equivalent to

$$\langle m'|S|n'\rangle = \langle m|S|n\rangle . \tag{2.10}$$

In order to explain the consequences of this assumption, let us consider a 2-to-2-particle scattering process, where the incoming (outgoing) particles have momenta p_1, p_2 (p_3, p_4). In a relativistic (1+1)-dimensional theory, energies and momenta of the particles involved in such scattering process can be conveniently encoded in a set of relativistic invariants, called Mandelstam variables [4]:

$$s = (p_1 + p_2)^2 , \quad t = (p_1 - p_3)^2 , \quad u = (p_1 - p_4)^2 , \tag{2.11}$$

where $p_i = (p_i^{(0)} = E_i, p_i^{(1)})$, such that $s + t + u = \sum_{i=1}^4 m_i^2$, due to the conservation law $p_1 + p_2 = p_3 + p_4$ and $p_i^2 = m_i^2$. Then the amplitude depends only on these Lorentz-invariant combinations of momenta, and in particular, since they are not independent, on two Mandelstam variables only.

[1] In this representation, both states and operators depend on time, then a generic physical state is defined as $|s_I(t)\rangle = e^{\mathrm{i}H_0 t}|s_S(t)\rangle$, where $|s_S(t)\rangle$ is the corresponding state in the Schrödinger picture. Then a generic operator in interaction picture is given in terms of the operator in Schrödinger representation by $O_I = O_I(t) = e^{\mathrm{i}H_0 t} O_S e^{-\mathrm{i}H_0 t}$.

Now, momenta and energies can be parametrized respectively as $p_i = m_i \sinh\theta_i$ and $E_i = m_i \cosh\theta_i$ in terms of the rapidity variable θ, while Mandelstam variables can be written as

$$s = m_1^2 + m_2^2 + 2m_1 m_2 \cosh(\theta_{12})\,, \tag{2.12}$$

$$t = m_1^2 + m_3^2 - 2m_1 m_3 \cosh(\theta_{13})\,, \tag{2.13}$$

$$u = m_1^2 + m_4^2 - 2m_1 m_4 \cosh(\theta_{14})\,, \tag{2.14}$$

where we introduced the notation $\theta_{ij} = \theta_i - \theta_j$. Then Lorentz invariance implies that the scattering phases depend only on the difference of the rapidities.
Another fundamental assumption is the so-called *macrocausality*, that play a fundamental rôle in the factorization property discussed in the next Section. Roughly speaking, macrocausality tells us that outgoing particles can propagate only once the interaction among the incoming ones happened, where "macro" means that this property can be violated on microscopic time scales. Finally, we will assume the *analyticity* of the S-matrices, namely they will be assumed to be analytic functions in the θ-plane with a minimal number of singularities dictated by specific physical processes.

3 Conserved charges and factorization

In a QFT, the notion of integrability is related to the existence of an infinite number of independent, conserved and mutually commuting charges Q_s. Then they can be diagonalized simultaneously:

$$Q_s|p\rangle_a = q_s^{(a)}(p)|p\rangle_a\,. \tag{3.1}$$

If they are local, *i.e.* they can be expressed as integrals of local densities, then they are additive:

$$Q_s|p_1,\dots,p_n\rangle_{a_1,\dots,a_n} = (q_s^{(a_1)}(p_1) + \cdots + q_s^{(a_n)}(p_n))|p_1,\dots,p_n\rangle_{a_1,\dots,a_n}\,. \tag{3.2}$$

Integrability has dramatic consequences on the form of the S-matrix: in $d > 2$ dimensions the Coleman-Mandula theorem [5] states that, even with a single charge being a second (or higher) order tensor, the theory has a trivial S-matrix: $S = 1$.
In (1+1) dimensions, instead, S-matrices do not trivializes. However, integrability is still very constraining and in particular we show that it implies

1. *no particle production*;
2. *final set of momenta = initial set of momenta*;
3. *factorization.*

Points 1. and 2. can be understood as follows. If a charge Q_s is conserved, then an initial eigenstate of Q_s with a given eigenvalue must evolve into a superposition of states sharing the same eigenvalue:

$$\sum_{i\in in}^{n} q_s^{(a_i)}(p_i) = \sum_{j\in out}^{m} q_s^{(b_j)}(p'_j)\,. \tag{3.3}$$

Since we have an infinite sequence of such constraints, these imply that $n = m$ and $p_i = p'_j$ ($q^{(s)}_{a_i} = q^{(s)}_{b_i}, i = 1, \dots, n$), namely the number of particles is the same before and after scattering and the initial and final sets of momenta are equal: in a word, the scattering is *elastic*.

3.1 Factorization and Yang-Baxter equation

In order to show point 3., that is the factorization of n-particle scattering into a product of 2-particle events

$$S_n(p_1, \dots, p_n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} S_2(p_i, p_j)\,, \tag{3.4}$$

we begin by an heuristic argument due to Zamolodchikov and Zamolodchikov [11].
Let us consider an n-particle configuration space ($\mathbb{R}^n$), with particles interacting at short range R. Then it is possible to consider $n!$ disconnected domains where the particles, with a permutation σ of ordered coordinates $x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_n}$ and momenta $p_{\sigma_1} > p_{\sigma_2} > \dots > p_{\sigma_n}$, are very far apart ($|x_{\sigma_{i+1}} - x_{\sigma_i}| \gg R$), so that they can be considered free.
Because of points 1. and 2., the wave function describing the particles in any single domain is a superposition of a finite number of n-particle plane waves:

$$\psi_\sigma(x_1, \dots, x_n) = \sum_{\sigma'} c(\sigma, \sigma') \exp[\mathrm{i}(p_{\sigma'_1} x_{\sigma_1} + \dots + p_{\sigma'_n} x_{\sigma_n})]\,, \tag{3.5}$$

with σ, σ' being permutations of $p_1, \dots, p_n$ allowed by the conditions of no particle production and conservation of momenta: basically, the set of momenta can only be reshuffled by scattering.
Since we assumed the existence of an asymptotic region (of free motion) for any permutation of particles, then the scattering process can be thought as a plane wave propagating from one of these asymptotic regions to another by passing through boundary interaction regions. Thus the propagation path can always be chosen in a way such that it goes through interaction regions where only two particles are so close to interact. For example, let us take those two particle being the particle 1 and 2, then such region is identified by

$$|x_1 - x_2| \ll R\,, \quad |x_1 - x_j| \gg R\,, \quad |x_2 - x_j| \gg R\,, \quad |x_i - x_j| \gg R\,, \quad i, j = 3, 4, \dots\,. \tag{3.6}$$

In this way only one particle at a time can overtake another, until all the particles starting from the configuration $(x_1, p_1), \dots, (x_n, p_n)$ have overtaken each other, to reach the configuration $(x_1, p_n), \dots, (x_n, p_1)$. All the other possible choices of paths connecting the same initial and final configurations, passing also through boundary regions with more than two interacting particles, have to give the same final result for the total scattering amplitude.
Not completely satisfied by this heuristic proof, we want to discuss a more rigorous argument, that dates back to [26] and [27]. For the reader interested in the details of the demonstration we refer to those papers and to [28], while what follows is mainly inspired by the review [10]. Demonstrations based on different approaches are given in [29] and, using non-local charges[2], in [30].

[2]See [31] for a definition of those charges on the basis of [30].

Let us start by considering a wave packet

$$\psi(x) = \int_{-\infty}^{+\infty} dp \exp[-a^2(p-p_0)^2] \exp[\mathrm{i}p(x-x_0)] \,, \tag{3.7}$$

with position and momentum centered around x_0 and p_0 respectively. We act on $\psi(x)$ with an operator $e^{\mathrm{i}cQ_s}$, where c is an arbitrary constant and Q_s is a conserved tensor of order s. The resulting wave function is given by

$$\tilde{\psi}(x) = \int_{-\infty}^{+\infty} dp \exp[-a^2(p-p_0)^2] \exp[\mathrm{i}p(x-x_0)] e^{\mathrm{i}cp^s} \,, \tag{3.8}$$

i.e. $e^{\mathrm{i}cQ_s}|p\rangle = e^{\mathrm{i}cp^s}|p\rangle$, since under a Lorentz transformation Q_s transforms as s copies of the total momentum $P = Q_1$.
Now, the wave packet is localized at a new position $x = x_0 - scp_0^{s-1}$, that is where the new phase is stationary ($\phi'(p_0) = 0$ with $\phi(p) = -a^2(p-p_0)^2 + \mathrm{i}p(x-x_0) + \mathrm{i}cp^s$). Thus the charge with $s = 1$, the total momentum, translates all the particles by the same amount c. In the case $s > 1$, instead, particles with different momenta are displaced by different amounts. In what follows, actually, we only need a couple of conserved charges Q_s, Q_{-s}, with $s > 1$ [28].
Let us then consider a scattering process with 2 incoming and m outgoing particles: the related scattering amplitude is

$$_{a_3,\dots,a_{m+2}}\langle p_3, \dots, p_{m+2}|S|p_1, p_2\rangle_{a_1,a_2} \,, \tag{3.9}$$

where the momenta are ordered as $p_1 > p_2; p_3 > p_4 > \cdots > p_{m+2}$. Now, the assumption of *macrocausality* for the S-matrix essentially tells us that the scattering amplitude is nonzero only if the outgoing particles are created after the incoming ones. In other words, the time t_{12} when the incoming particle 1 collide with particle 2 has to be smaller than the time t_{23} when the slowest incoming particle (particle 2) interacts with the fastest outgoing particle (particle 3): $t_{23} \geq t_{12}$.
Since the charge Q_s commutes with the S-matrix, we can use it to rearrange initial and final configurations without changing the amplitude:

$$_{a_3,\dots,a_{m+2}}\langle p_3, \dots, p_{m+2}|S|p_1, p_2\rangle_{a_1,a_2} = {}_{a_3,\dots,a_{m+2}}\langle p_3, \dots, p_{m+2}|e^{-\mathrm{i}cQ_s} S e^{\mathrm{i}cQ_s}|p_1, p_2\rangle_{a_1,a_2} \,. \tag{3.10}$$

This means that, with a suitable choice of c, t_{23} can be made smaller than t_{12}, and, if any of the outgoing particles is different from the incoming ones, then the amplitude vanishes, following the macrocausality principle.
Therefore the only possibility is that one has just 2 outgoing particles with the same momenta p_1, p_2 as the incoming ones. With this we showed that the scattering has to be *elastic*.
In order to prove the *factorization*, we have to consider processes with more than 2 particles. In this case, we know now that, acting with a charge like in (3.10), we can separate as much as we want the trajectories of the particles without changing the resulting amplitude, and

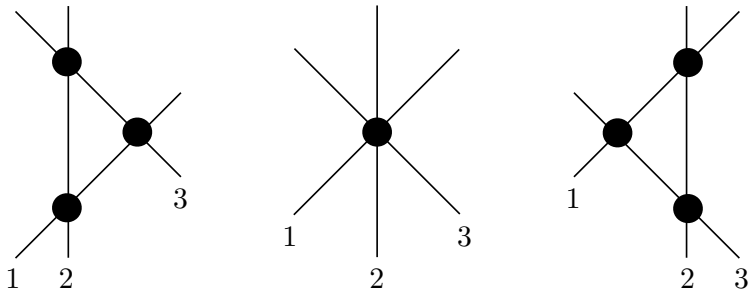

Figure 1: 3-particle scattering amplitudes.

then also the points of interaction between couples of particles (which we know now can produce only couples of particle with momenta equal to the incoming ones): then the total scattering can happen as a sequence of 2-particle interactions.
In other words, considering the 3-particle example, the three types of possible collisions shown in Figure 1 can be obtained one from each other by suitable actions of e^{icQ_s} with different values of c; all of them, anyway, commute with the Hamiltonian and the S-matrix, then they have to give physically equivalent processes.
This equivalence is formalized in the famous *Yang-Baxter equation* (YBE) [32, 33]:

$$S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}\,, \tag{3.11}$$

where for simplicity we labeled the S-matrices just by the labels of the particles of kind $1, 2, 3$ involved in a three-particle process. We can write (3.11) in components in order to show the matrix elements involved in a generic non-diagonal process, where exchanges of flavors among particles are possible, in the following way (see also Figure 2):

$$\sum_{c_1,c_2,c_3} S^{c_1c_2}_{a_1a_2}(\theta_{12})S^{b_1c_3}_{c_1a_3}(\theta_{13})S^{b_2b_3}_{c_2c_3}(\theta_{23}) = \sum_{c_1,c_2,c_3} S^{c_2c_3}_{a_2a_3}(\theta_{23})S^{c_1b_3}_{a_1c_3}(\theta_{13})S^{b_1b_2}_{c_1c_2}(\theta_{12})\,. \tag{3.12}$$

The generalization to n-particle is straightforward. A 4-particle process can be always separated in a 3-particle one, for which the YBE (3.12) is already shown, and three 2-particle processes, by displacing a particle. Then the YBE is proven for 4-particle processes. In the same way one decomposes a 5-particle scattering in processes involving at most four particles, and so on.
Now we can understand better why in $d > 2$ the S-matrix of an integrable theory must be trivial: essentially, in $d > 2$ it is always possible to move the trajectories of the particles to create equivalent scattering processes where particles are not crossing each other.

4 2-particle S-matrix

From the discussion of the previous Section, it turns out that any n-particle scattering process in integrable theories is completely determined by the knowledge of the 2-particle S-matrix. Therefore, in this Section, we will focus on general physical properties and the analytic structure of 2-particle S-matrices.

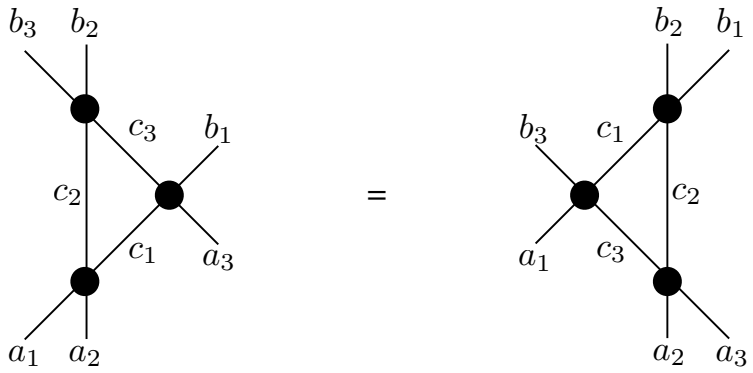

Figure 2: Yang-Baxter equation.

4.1 Properties and analytic structure

Following the general definition (2.4), in the case of a 2-particle *elastic scattering* with incoming (outgoing) rapidities θ_1, θ_2 (θ_3, θ_4) we have $\theta_1 = \theta_4, \theta_2 = \theta_3$ and $S = S(\theta_1 - \theta_2)$. A 2-particle elastic relativistic S-matrix is then given by

$$|\theta_1, \theta_2\rangle^{in}_{i,j} = S^{kl}_{ij}(\theta_1 - \theta_2)\, |\theta_1, \theta_2\rangle^{out}_{k,l} \ , \tag{4.1}$$

with $\theta_1 > \theta_2$, and represented graphically in Figure 3. In terms of Mandelstam variables, $u = 0$ and $t(\theta_{12}) = s(\mathrm{i}\pi - \theta_{12})$, then the S-matrix depends only on one variable, say $S = S(s)$.

Now, we want to answer the question: how to determine the 2-particle S-matrix elements? Let us begin from the constraints given by *discrete symmetries* usually respected by physical QFTs. If the theory is invariant under reflection of space coordinates, *i.e.* under *parity*, it means that looking at Figure 3 from left to right or from right to left has to be equivalent. Namely, the particles i and k can be exchanged with j and l respectively, leaving the amplitudes unchanged:

$$S^{kl}_{ij}(\theta) = S^{lk}_{ji}(\theta)\,. \tag{4.2}$$

Analogously, the symmetry under *time reversal* implies that the amplitude represented in Figure 3 is the same if we look at it from bottom to top or vice versa, then by exchanging particles i and l, j and k:

$$S^{kl}_{ij}(\theta) = S^{ji}_{lk}(\theta)\,. \tag{4.3}$$

If a theory is invariant under *charge conjugation*, then we require that the S-matrix does not change under conjugation of the particles involved in the scattering process:

$$S^{\bar{k}\bar{l}}_{\bar{i}\bar{j}}(\theta) = S^{kl}_{ij}(\theta)\,, \tag{4.4}$$

where we denoted the charge-conjugated particles by barred indices.

Now, in order to study the *analytic properties* of the S-matrix, we recall the definitions (2.11) of the Mandelstam variables. It is easy to understand that s, t and u are the center-of-mass squared energies in the channels defined by the process $i + j \to k + l$ (s-channel),

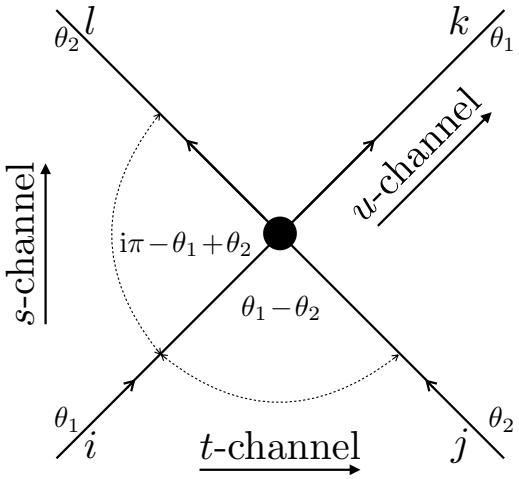

Figure 3: 2-particle elastic S-matrix element $S_{ij}^{kl}(\theta_1 - \theta_2)$.

$i + \bar{l} \to k + \bar{j}$ (t-channel) and $i + \bar{k} \to l + \bar{j}$ (u-channel) respectively, as depicted in Figure 3:

$$s = (E_i + E_j)^2\,, \quad t = (E_i + E_{\bar{l}})^2\,, \quad u = (E_i + E_{\bar{k}})^2\,. \tag{4.5}$$

In a physical process, $\theta_i - \theta_j$ has to be real, then s has to be in the so-called *physical region*, defined by $s^+ = s + \mathrm{i}0$ and $s \geq (m_i + m_j)^2$, *i.e.* slightly above the right cut in the first of Figure 4 (a).

Then let us study the analytical continuation of $S(s)$ to the s-plane. We begin by imposing *unitarity* in the physical region:

$$S_{ij}^{kl}(s^+)(S_{lk}^{mn})^*(s^+) = \delta_i^n \delta_j^m\,. \tag{4.6}$$

According to the *analyticity* assumption, the S-matrix in the physical region is the boundary value of a function that is analytic in the whole s-plane, then the unitarity property (4.6) can be extended to the so-called *hermitian analyticity*:

$$S_{ij}^{kl}(s^*) = (S_{kl}^{ij})^*(s)\,. \tag{4.7}$$

Adding to this property the time reversal symmetry, we get a stronger condition, that is the *real analyticity*[3]:

$$S_{ij}^{kl}(s^*) = (S_{ij}^{kl})^*(s)\,, \tag{4.8}$$

i.e. the S-matrix is real for real values of s and $(m_i - m_j)^2 \leq s \leq (m_i + m_j)^2$. This means, in general, that real S-matrices do not describe physical processes.

Another fundamental property constraining relativistic S-matrices is the *crossing symmetry*, meaning that the process in Figure 3 has to be read equivalently along the s- and t-channels:

$$S_{ij}^{kl}(s) = S_{\bar{l}i}^{\bar{j}k}(t)\,, \tag{4.9}$$

[3]See an interesting discussion on hermitian and real analyticity of the S-matrix, and their interplay, in [34] and references therein.

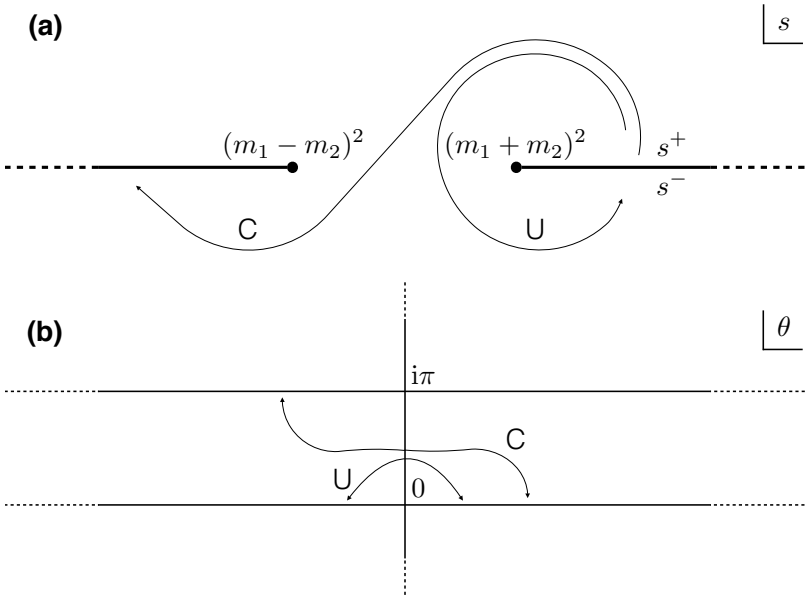

Figure 4: S-matrix analytical properties in the s-plane (a) and θ-plane (b). U and C stand for unitarity and crossing transformations, respectively.

where, as in (4.4), barred indices denote charge-conjugated particles or anti-particles, that can be considered also as particles propagating backwards in time. In terms of the rapidity, since $t(\theta) = s(\mathrm{i}\pi - \theta)$, crossing symmetry can be written as

$$S_{ij}^{kl}(\theta) = S_{\bar{l}i}^{\bar{j}k}(\mathrm{i}\pi - \theta)\,. \tag{4.10}$$

In particular, denoting by $\mathcal{C}$ the charge-conjugation operator, crossing symmetry can be written also as

$$S_{ij}^{kl}(\theta) = \mathcal{C}_{jn} S_{mi}^{nk}(\mathrm{i}\pi - \theta)\mathcal{C}^{ml}\,. \tag{4.11}$$

Note that relation (4.11) involves a dynamical transformation - in contrast to unitarity or discrete symmetries, where only the matrix form is involved - on the rapidity. As we will see in Section 4.3 and in the examples of Section 8, crossing symmetry plays a fundamental rôle in fixing the scalar factors of the S-matrices. It is a property that profoundly reflects the relativistic invariance of the theory, since it uses the invariance of physical processes under exchange of space and time, *i.e.* under rotation of the s-channel to the t-channel. However, it is possible to generalise crossing symmetry to non-relativistic theories like AdS/CFT thanks to its formulation in completely algebraic ways [17, 19], that will be discussed in Sections 4.2 and 6.

Turning back to the relativistic case, we notice that real analyticity (4.8) entails

$$S_{ij}^{kl}(s^+)(S_{lk}^{mn})(s^-) = \delta_i^n \delta_l^m\,, \tag{4.12}$$

where $s^- = s - \mathrm{i}0$. Equation (4.12) means that the S-matrix[4] has a branch cut between

[4]Except for the trivial cases of $S = \pm 1$.

the regions where s^+ and s^- are respectively defined, namely there is a branch point in $s = (m_i + m_j)^2$. This is expected also since that point corresponds to the two-particle threshold, *i.e.* it is a discontinuity point of the amplitude imaginary part (see more details on this aspect in [9]). Because of crossing symmetry, another branch cut starting from $s = (m_i - m_j)^2$ towards $-\infty$ must exist, as depicted in Figure 4 (a)[5]. These are the only two branch cuts if the S-matrix is factorized, since particle production thresholds for more than two particles cannot appear.
Moreover, it is possible to show that the branch cut is of square root type, since unitarity gives

$$S(s^+)S_\gamma(s^+) = 1\,, \tag{4.13}$$

where S_γ is the S-matrix analytically continued below the cut around the branch point $(m_i + m_j)^2$, and then

$$S_\gamma(s^-) = S^{-1}(s^-) = S(s^+)\,, \tag{4.14}$$

where we used the real analyticity (4.12). The last relation means basically that a double continuation around the branch point gives back the original S-matrix, *i.e.* the branch cut is of square root type.
In order to show in a more concise way the analytical properties of the S-matrix, it is convenient to switch from the variable s to the difference of rapidities, via the map

$$\theta_1 - \theta_2 = \log\left(\frac{s - m_1^2 - m_2^2 + \sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{2m_1m_2}\right). \tag{4.15}$$

Then the *physical sheet* maps to the strip $0 \leq \mathrm{Im}(\theta_1 - \theta_2) \leq \pi$, the second sheet corresponds to $\pi \leq \mathrm{Im}(\theta_1 - \theta_2) \leq 2\pi$ and so on, with periodicity 2πi. Essentially, the branch cuts of the s-plane open up in such a way that all the Riemann sheets are mapped into strips $n\pi \leq \mathrm{Im}(\theta) \leq (n+1)\pi$ and S is analytic at the images i$n\pi$ of the branch points. In conclusion, $S(\theta)$ is a meromorphic function of θ and its real analyticity implies that it is real on the imaginary axis of θ. The main analytic properties of the 2-particle relativistic S-matrix can be represented in the θ-plane as in Figure 4 (b).

4.2 Zamolodchikov-Faddeev algebra

After having discussed the analytic properties of the 2-particle integrable S-matrix, let us move to its algebraic features. To do this, we introduce a purely algebraic setup, that is fully consistent with the properties studied in the previous Section and will make easier the understanding of the algebraic structures discussed in Section 6. It will be also useful to extend some properties to the non-relativistic case, as explained in Section 4.2.1.
Let us start by defining the creation and annihilation operators of excitations out of the vacuum state $|0\rangle$, that is left invariant by the symmetry algebra of the particular integrable quantum model under study:

$$\bar{A}^{a_j}(p_j)|0\rangle = 0 = \langle 0|A_{a_j}(p_j)\,. \tag{4.16}$$

[5] A different choice of the branch cuts is not equivalent, since $s = \pm\infty$ are branch points too.

In particular, the particles created by $|p_j\rangle_{a_j} = A_{a_j}(p_j)|0\rangle$ have momenta p_i and transform in a linear irreducible representation of the symmetry algebra.
Then the asymptotic states (2.2) can be written as

$$|p_1, p_2, \dots, p_n\rangle^{in}_{a_1,a_2,\dots,a_n} = A_{a_1}(p_1)A_{a_2}(p_2)\dots A_{a_n}(p_n)|0\rangle\,, \tag{4.17}$$
$$|p_1, p_2, \dots, p_n\rangle^{out}_{a_1,a_2,\dots,a_n} = A_{a_n}(p_n)\dots A_{a_2}(p_2)A_{a_1}(p_1)|0\rangle\,, \tag{4.18}$$

with $p_1 > p_2 > \dots > p_n$. On the other hand, the conjugated operators $\bar{A}_{a_j}(p_j)$ generate the dual states

$$\overset{in}{}_{a_1,a_2,\dots,a_n}\langle p_1, p_2, \dots, p_n| = \langle 0|\bar{A}^{a_1}(p_1)\bar{A}^{a_2}(p_2)\dots\bar{A}^{a_n}(p_n)\,, \tag{4.19}$$
$$\overset{out}{}_{a_1,a_2,\dots,a_n}\langle p_1, p_2, \dots, p_n| = \langle 0|\bar{A}^{a_n}(p_n)\dots\bar{A}^{a_2}(p_2)\bar{A}^{a_1}(p_1)\,. \tag{4.20}$$

The operators $A_{a_j}(p_j), \bar{A}^{a_j}(p_j)$ are elements of an associative non-commutative algebra, the so-called Zamolodchikov-Faddeev (ZF) algebra [11, 35]. In a relativistic case, (4.17)-(4.20) can be conveniently parametrised by the particles rapidities:

$$|\theta_1, \theta_2, \dots, \theta_n\rangle^{in}_{a_1,a_2,\dots,a_n} = A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)|0\rangle\,, \tag{4.21}$$
$$|\theta_1, \theta_2, \dots, \theta_n\rangle^{out}_{a_1,a_2,\dots,a_n} = A_{a_n}(\theta_n)\dots A_{a_2}(\theta_2)A_{a_1}(\theta_1)|0\rangle\,, \tag{4.22}$$
$$\overset{in}{}_{a_1,a_2,\dots,a_n}\langle \theta_1, \theta_2, \dots, \theta_n| = \langle 0|\bar{A}^{a_1}(\theta_1)\bar{A}^{a_2}(\theta_2)\dots\bar{A}^{a_n}(\theta_n)\,, \tag{4.23}$$
$$\overset{out}{}_{a_1,a_2,\dots,a_n}\langle \theta_1, \theta_2, \dots, \theta_n| = \langle 0|\bar{A}^{a_n}(\theta_n)\dots\bar{A}^{a_2}(\theta_2)\bar{A}^{a_1}(\theta_1)\,. \tag{4.24}$$

For simplicity of notation, all the following equations involving ZF operators will be understood as acting on $|0\rangle$.
Defining the asymptotic states in this way allows to interpret the scattering processes as simple reordering of ZF operators in the rapidity space. Indeed, writing explicitly the asymptotic states of equation (4.1) in terms of ZF generators as in (4.21), (4.22) and dropping the vacuum states, it becomes

$$A_i(\theta_1)A_j(\theta_2) = A_l(\theta_2)A_k(\theta_1)S^{kl}_{ij}(\theta_1 - \theta_2)\,, \tag{4.25}$$

that is the commutation relation between the ZF algebra elements, and it can be interpreted as definition of the 2-particle S-matrix. The ZF algebra is completed by the commutation relations involving the annihilation operators (4.16):

$$\bar{A}^i(\theta_1)\bar{A}^j(\theta_2) = S^{ij}_{kl}(\theta_1 - \theta_2)\bar{A}^l(\theta_2)\bar{A}^k(\theta_1)\,, \tag{4.26}$$
$$\bar{A}^k(\theta_1)A_j(\theta_2) = A_l(\theta_2)S^{kl}_{ij}(\theta_2 - \theta_1)\bar{A}^i(\theta_1) + \delta(\theta_1 - \theta_2)\delta^k_j\,, \tag{4.27}$$

that generalize the usual bosonic and fermionic canonical commutation relations, corresponding to $S = 1$ and $S = -1$ respectively. The δ-function in the r.h.s of (4.27) is related to the normalization of the states, that is ${}_i\langle\theta_1|\theta_2\rangle_j = \delta(\theta_1 - \theta_2)\delta_{ij}$.
Now, writing the commutation relation for the elements labeled by k and l

$$A_l(\theta_2)A_k(\theta_1) = S^{mn}_{lk}(\theta_2 - \theta_1)A_n(\theta_1)A_m(\theta_2)\,, \tag{4.28}$$

and plugging it into (4.25), one can show

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1-\theta_2)S_{lk}^{mn}(\theta_2-\theta_1)A_n(\theta_1)A_m(\theta_2)\,, \tag{4.29}$$

that is equivalent to

$$S_{ij}^{kl}(\theta_1-\theta_2)S_{lk}^{mn}(\theta_2-\theta_1) = \delta_i^n\delta_j^m\,. \tag{4.30}$$

This property is also called *braiding unitarity*.
On the other hand, in order to get (4.6), also referred as *physical unitarity*, we have to take the hermitian conjugation of (4.25):

$$\bar{A}^j(\theta_2)\bar{A}^i(\theta_1) = (S^\dagger)_{kl}^{ij}(\theta_1-\theta_2)\bar{A}^k(\theta_1)\bar{A}^l(\theta_2)\,. \tag{4.31}$$

Thus, exchanging θ_1 with θ_2 and permuting the ZF operators, we get

$$\bar{A}^i(\theta_1)\bar{A}^j(\theta_2) = (S^\dagger)_{kl}^{ij}(\theta_2-\theta_1)\bar{A}^l(\theta_2)\bar{A}^k(\theta_1)\,. \tag{4.32}$$

But we also know that (4.26) holds, then $S_{ij}^{kl}(\theta_1-\theta_2) = (S^\dagger)_{ij}^{kl}(\theta_2-\theta_1)$. Finally, using the braiding unitarity (4.30), we get (4.6): $SS^\dagger = 1$.

Exercises

1. We leave as an exercise the derivation of CPT invariance using the ZF algebra and knowing that $A_i(\theta) \to A_i(-\theta)$ under parity and time reversal. The charge-conjugation symmetry, on the other hand, requires that the ZF algebra maps to itself under the transformations $A_i(\theta) \to \bar{A}_i^t(\mathrm{i}\pi+\theta)\mathcal{C}$, $\bar{A}_i(\theta) \to \mathcal{C}^\dagger A_i^t(\mathrm{i}\pi+\theta)$, where $\mathcal{C}$ is the charge-conjugation matrix defined by $A_i(\theta) = \mathcal{C}_{ij}\bar{A}^j$ and the superscript t denotes the transposition.

2. Prove that, if the charge conjugation acts only on one sector of the 2-particle space, one gets the crossing symmetry relation (4.11).

3. Show that the associativity property of the ZF algebra implies the YBE (3.11).

4.2.1 Non-relativistic case

In a non-relativistic model, the S-matrix does not depend on the difference of rapidities, but separately on the momenta of the particles. Therefore, the ZF algebra generalizes to

$$\begin{aligned}
A_i(p_1)A_j(p_2) &= A_l(p_2)A_k(p_1)S_{ij}^{kl}(p_1,p_2)\,, &(4.33)\\
\bar{A}^i(p_1)\bar{A}^j(p_2) &= S_{kl}^{ij}(p_1,p_2)\bar{A}^l(p_2)\bar{A}^k(p_1)\,, &(4.34)\\
\bar{A}^k(p_1)A_j(p_2) &= A_l(p_2)S_{ij}^{kl}(p_2,p_1)\bar{A}^i(p_1) + \delta(p_1-p_2)\delta_j^k\,. &(4.35)
\end{aligned}$$

Analogously, the YBE (3.12) becomes

$$\sum_{c_1,c_2,c_3} S_{a_1a_2}^{c_1c_2}(p_1,p_2)S_{c_1a_3}^{b_1c_3}(p_1,p_3)S_{c_2c_3}^{b_2b_3}(p_2,p_3) = \sum_{c_1,c_2,c_3} S_{a_2a_3}^{c_2c_3}(p_2,p_3)S_{a_1c_3}^{c_1b_3}(p_1,p_3)S_{c_1c_2}^{b_1b_2}(p_1,p_2)\,, \tag{4.36}$$

and the physical properties discussed in Section 4.1 can be derived using the properties of the ZF algebra. For example, relations similar to (4.28) and (4.29) lead to the braiding unitarity condition

$$S_{ij}^{kl}(p_1,p_2)S_{lk}^{mn}(p_2,p_1) = \delta_i^n\delta_j^m\,. \tag{4.37}$$

Together with relations analogous to (4.31) and (4.32), (4.37) gives the physical unitarity condition

$$(S^\dagger)_{ij}^{kl}(p_1,p_2)S_{kl}^{mn}(p_1,p_2) = \delta_i^m\delta_j^n\,. \tag{4.38}$$

Furthermore, the properties of the asymptotic states under transformations of parity and time reversal, respectively denoted by $\mathcal{P}$ and $\mathcal{T}$,

$$\mathcal{P}|p_1,p_2,\dots,p_n\rangle_{i_1,\dots,i_n}^{(in)} = |-p_1,-p_2,\dots,-p_n\rangle_{i_1,\dots,i_n}^{(in)}\,, \tag{4.39}$$

$$\mathcal{T}|p_1,p_2,\dots,p_n\rangle_{i_1,\dots,i_n}^{(in)} = |-p_1,-p_2,\dots,-p_n\rangle_{i_1,\dots,i_n}^{(out)}\,, \tag{4.40}$$

written in terms of ZF operators as

$$\mathcal{P}A_{i_1}(p_1)\dots A_{i_n}(p_n)|0\rangle = A_{i_n}(-p_n)\dots A_{i_1}(-p_1)|0\rangle\,, \tag{4.41}$$

$$\mathcal{T}A_{i_1}(p_1)\dots A_{i_n}(p_n)|0\rangle = A_{i_1}(-p_1)\dots A_{i_n}(-p_n)|0\rangle\,, \tag{4.42}$$

allow us to generalize the discrete symmetries listed in Section 4.1 for the relativistic case in the following way (see Chapter 3 of [12] for further details on the derivation):

- *parity*: $S_{ij}^{kl}(p_1,p_2) = S_{ji}^{lk}(-p_2,-p_1)$,
- *time reversal*: $S_{ij}^{kl}(p_1,p_2) = S_{lk}^{ji}(-p_2,-p_1)$,

while the symmetry under *charge conjugation* translates trivially to the condition

$$S_{ij}^{kl}(p_1,p_2) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(p_1,p_2)\,, \tag{4.43}$$

or, using the charge-conjugation operator $\mathcal{C}$,

$$S_{ij}^{kl}(p_1,p_2) = \mathcal{C}_{ir}\mathcal{C}_{js}S_{mn}^{rs}(p_1,p_2)\mathcal{C}^{mk}\mathcal{C}^{nl}\,, \tag{4.44}$$

where $\mathcal{C}_{ij}\mathcal{C}^{jk} = \delta_i^k$.

Although *crossing symmetry* is a property that emerges naturally in the context of relativistic scattering theories and at a first approach its generalization to systems where time and space cannot be exchanged might seem impossible, it can be recovered, as all the other properties discussed above, from an additional requirement on the ZF algebra [12,19]. Basically, we recall that in the relativistic case the crossing transformation entails an exchanging of a particle with an anti-particle and a kinematical map $\theta \to \mathrm{i}\pi + \theta$ on the

rapidity of the conjugated particle. This translates to the maps $p \to -p$ and $E \to -E$ on the momentum and energy of a non-relativistic particle.
Then the ZF generators must transform as

$$A_i(p) \to A_{\bar{i}}(-p) = A_j(-p)\mathcal{C}^{ji} \; ; \quad \bar{A}^i(p) \to \bar{A}^{\bar{i}}(-p) = \mathcal{C}_{ij}A^j(-p)\,. \tag{4.45}$$

Requiring that the commutation relations (4.33)-(4.35) are invariant under this transformation implies

$$S_{ij}^{kl}(p_1,-p_2) = \mathcal{C}_{jn}S_{mi}^{nk}(p_1,p_2)\mathcal{C}^{ml}\,, \tag{4.46}$$
$$S_{ij}^{kl}(-p_1,p_2) = \mathcal{C}_{in}S_{jm}^{ln}(p_1,p_2)\mathcal{C}^{mk}\,, \tag{4.47}$$

that are the crossing symmetry relations for a non-relativistic 2-particle S-matrix.

4.3 General relativistic solutions

Turning back to relativistic S-matrices, we want to show here how they can be completely determined using their analytic properties and symmetries. First of all, the YBE can determine the ratios between S-matrix elements that belong to the same mass multiplet. Thus, a general solution of the YBE can be written as

$$S_{ij}^{kl}(\theta) = \frac{1}{f(\theta)}R_{ij}^{kl}(\theta)\,, \tag{4.48}$$

where R is the matrix of the ratios between amplitudes fixed by the YBE, f and R_{ij}^{kl} are meromorphic functions of θ.

Exercises

1. Show that R satisfies $R_{ij}^{kl}(0) = \delta_i^l\delta_j^k R_0$ (from the commutation relation of the ZF algebra).

2. Using the previous relation and the YBE (3.12), show that

$$R_{ij}^{nm}(\theta)R_{nm}^{kl}(-\theta) = \delta_i^k\delta_j^l Q(\theta)\,, \tag{4.49}$$

and that the braiding unitarity reduces to

$$f(\theta)f(-\theta) = Q(\theta)\,. \tag{4.50}$$

Rescaling the rapidity by an arbitrary constant λ ($\theta \to \lambda\theta$), R is still solution of the YBE and, for a suitable choice of λ, in all the known cases it satisfies

$$R_{ij}^{kl}(\theta) = R_{il}^{kj}(\mathrm{i}\pi - \theta)\,. \tag{4.51}$$

Then the crossing symmetry reduces to

$$f(\theta) = f(\mathrm{i}\pi - \theta)\,. \tag{4.52}$$

Therefore $f(\theta)$ is fixed by (4.50) and (4.52) up to a function $\phi(\theta)$, called CDD factor [36], satisfying

$$\phi(\theta) = \phi(\mathrm{i}\pi - \theta)\,, \tag{4.53}$$

$$\phi(\theta)\phi(-\theta) = 1\,. \tag{4.54}$$

This ambiguity corresponds to the freedom to add zeros and poles with period $2\pi\mathrm{i}$ to $f(\theta)$, due to the infinite discrete set of solutions for ϕ, in general. So, if we denote as $f_{min}(\theta)$ the solution of (4.50) and (4.52) with minimal number of poles and zeros, then the general solution for f is $f(\theta) = f_{min}(\theta)\phi(\theta)$.
Another fundamental restriction for generic S-matrix elements is the invariance under the symmetry algebra of the model under study. The corresponding constraints can be derived by acting with the symmetry generators J^a, where a runs from 1 to the dimension of the symmetry algebra, on the ZF relations (4.25):

$$J^a A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_1 - \theta_2)J^a A_l(\theta_2)A_k(\theta_1)\,. \tag{4.55}$$

The action of J^a on the 2-particle states is given by

$$J^a A_i(\theta_1)A_j(\theta_2) = (J^a)_{ij}^{kl}(\theta_1, \theta_2)A_k(\theta_1)A_l(\theta_2)\,, \tag{4.56}$$

where $(J^a)_{ij}^{kl}$ are the matrix elements of the 2-particle generator J_{12}^a, that acts on the 2-particle spaces as $J_{12}^a = J^a \otimes \mathbb{I} + \mathbb{I} \otimes J^a$. Thus, the S-matrix invariance can be written in matrix form as

$$(J^a \otimes \mathbb{I} + \mathbb{I} \otimes J^a)S = S(J^a \otimes \mathbb{I} + \mathbb{I} \otimes J^a)\,. \tag{4.57}$$

Summarizing, the steps necessary to compute the S-matrix in an integrable theory are the following:

- determine the structure of the S-matrix by imposing invariance under the symmetry generators (by solving the equations given by the condition (4.57));
- find the ratios between the remaining undetermined S-matrix elements by imposing the YBE (3.11);
- fix the remaining (minimal) overall scalar factor, up to CDD factors, by imposing unitarity and crossing symmetry.

We will see some detailed application of this algorithm in few particular cases (sine-Gordon, $SU(2)$ and $SU(3)$ chiral Gross-Neveu models) discussed in Section 8.
In the non-relativistic example of AdS_5/CFT_4, for instance, the S-matrix for the fundamental excitations was determined in [16], up to a scalar factor, imposing invariance under two copies of centrally extended $SU(2|2)$ symmetry algebras. Such S-matrix turned out to

satisfy identically the YBE, while the crossing symmetry condition, implemented in [19] and [17] through the algebraic frameworks illustrated respectively in the previous Section and in Section 6, led to an equation for the scalar factor, that was solved in [37] (see also the review [14]).

4.3.1 Purely elastic case

While elastic scattering essentially means that the set of outgoing particles is identical to the incoming one, *purely elastic scattering* is further constrained by not having reflection between particles. So, particles can be just transmitted and the S-matrix is *diagonal*:

$$S_{ij}^{kl} = \delta_i^k \delta_j^l S_{ij} \,. \tag{4.58}$$

The YBE is identically satisfied and the system of equations of unitarity and crossing symmetry is solved by a function $S(\theta)$ with period $2\pi\mathrm{i}$, given by [38]

$$S(\theta) = \prod_\alpha f_\alpha(\theta)\,; \quad f_\alpha(\theta) = \frac{\sinh\left(\frac{\theta+\mathrm{i}\pi\alpha}{2}\right)}{\sinh\left(\frac{\theta-\mathrm{i}\pi\alpha}{2}\right)}\,, \tag{4.59}$$

with α belonging to a subset A_{ij} of $\mathbb{C}$ invariant under complex conjugation. Indeed one can easily verify that

$$f_\alpha(\theta) f_{\bar\alpha}(\theta) = f_\alpha(2\pi\mathrm{i}+\theta) f_{\bar\alpha}(2\pi\mathrm{i}+\theta)\,, \tag{4.60}$$

for any complex α. Periodicity implies that α can be chosen in the interval $-1 \le \alpha \le 1$. Poles are at $\mathrm{i}\pi\alpha$, while zeros are at $-\mathrm{i}\pi\alpha$, then they are contained in the strip $-\pi \le \mathrm{Im}(\theta) \le \pi$.
In case of neutral particles (particle = antiparticle), then $S_{ij}(\theta) = S_{ij}(\mathrm{i}\pi - \theta)$ and the solution of unitarity and crossing is a product over arbitrary α of the functions

$$F_\alpha(\theta) = f_\alpha(\theta) f_\alpha(\mathrm{i}\pi - \theta)\,, \tag{4.61}$$

with simple poles at $\theta = \mathrm{i}\pi\alpha, \mathrm{i}\pi(1-\alpha)$, zeros at $\theta = -\mathrm{i}\pi\alpha, -\mathrm{i}\pi(1-\alpha)$, related by crossing. Anyway, unitarity and crossing are not sufficient to fix the sets of poles/zeros A_{ij} of $\prod_{\alpha\in A_{ij}} f_\alpha$ or $\prod_{\alpha\in A_{ij}} F_\alpha$. We will see in the next Section how it is actually possible to fix them.

5 Poles structure and bootstrap principle

Since in the region $(m_i - m_j)^2 \le s \le (m_i + m_j)^2$ ($0 \le \mathrm{Im}(\theta_{ij}) \le \pi$ in terms of rapidity) it is possible to create, from incoming particle of masses m_i and m_j, only 1-particle states with $m < m_i + m_j$, then simple poles of the S-matrix in that range of s are generally expected to correspond to *bound states*. In order to clarify this correspondence with a simple example, let us consider the one-dimensional scattering problem associated to a quantum mechanical system with delta potential.

5.1 Delta potential scattering problem

We want to solve the Schrödinger problem corresponding to the Hamiltonian

$$H = \frac{p^2}{2m} + V(x)\,, \tag{5.1}$$

with potential $V(x) = -2g\delta(x)$ and $g > 0$. The Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - 2g\delta(x)\right)|\psi\rangle = E|\psi\rangle\,, \tag{5.2}$$

can be conveniently rewritten as

$$\left(-\frac{d^2}{dx^2} - 2g\delta(x)\right)|\psi\rangle = k^2|\psi\rangle\,, \tag{5.3}$$

by rescaling $g \to \frac{\hbar^2}{2m}g$ and defining $k^2 = E\frac{2m}{\hbar^2}$. We look for solutions of (5.3) in the following generic form

$$\psi(x) = \begin{cases} Ae^{\mathrm{i}kx} + Be^{-\mathrm{i}kx}\,; & x < 0\,, \\ Ce^{\mathrm{i}kx} + De^{-\mathrm{i}kx}\,; & x > 0\,. \end{cases} \tag{5.4}$$

Since we want to consider the scattering of incident particles coming from the left and being reflected or transmitted by the δ-potential barrier, then we have not incoming waves from the right, *i.e.* $D = 0$, and A, B, C are the amplitudes of the incoming, reflected and transmitted wave packets respectively. These coefficients can be found by solving the continuity condition of the wave function across the point $x = 0$

$$\psi(0^-) = \psi(0^+)\,, \tag{5.5}$$

and the discontinuity condition on the first derivative of $\psi(x)$ given by integrating equation (5.3) between ϵ and $-\epsilon$, with $\epsilon \to 0$

$$\psi'(0^-) - \psi'(0^+) - 2g\psi(0) = 0\,. \tag{5.6}$$

Condition (5.5) gives

$$A + B = C\,, \tag{5.7}$$

while (5.6) implies

$$\mathrm{i}kB + gC = 0\,. \tag{5.8}$$

Thus, the transmission and reflection coefficient are, respectively

$$T = \frac{C}{A} = \frac{k}{k - \mathrm{i}g}\,; \quad R = \frac{B}{A} = \frac{\mathrm{i}g}{k - \mathrm{i}g}\,. \tag{5.9}$$

If k is complex, its imaginary part contributes to the real parts of the exponentials in (5.4). Moreover, both the transmission and reflection coefficients in (5.4) have a pole in $k = \mathrm{i}g$, but we can still normalize the incoming wave function by setting $A = 0$. Thus, at the value $k = \mathrm{i}g$, with $g > 0$, (5.4) gives a physically admissible solution, *i.e.* decreasing to

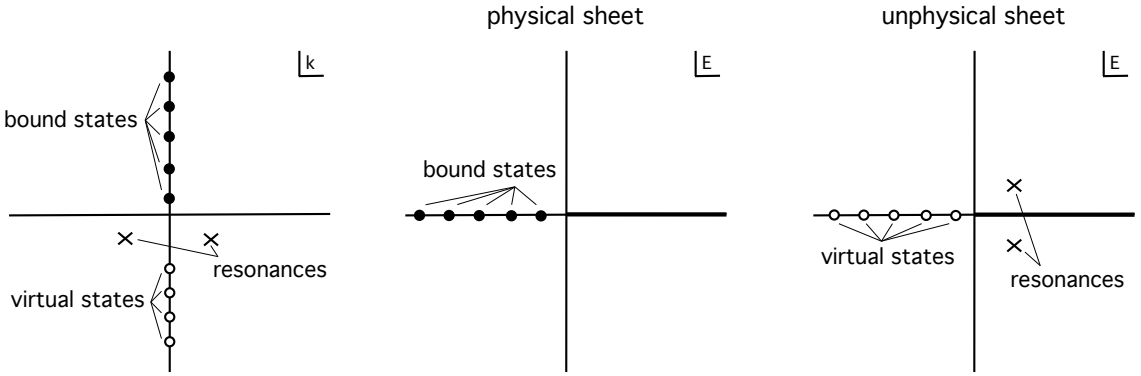

Figure 5: S-matrix poles on k- and E-plane.

zero at large distances, with just outgoing waves and not incoming ones: it corresponds to a bound state.
Moreover, considering the time evolution of (5.4)

$$\psi(x,t) = e^{-\mathrm{i}t\frac{\hbar k^2}{2m}}\psi(x) \tag{5.10}$$

we see that no solutions can exist with $k = k_1 + \mathrm{i}k_2$ ($k_{1,2} \in \mathbb{R}$), $k_1 \neq 0$ and $k_2 > 0$, since (5.10) would increase exponentially with time in some channel. This would contradict the conservation of probability, then there are no poles of the S-matrix with non-vanishing real part in the upper half plane of k.
Poles of the S-matrix with negative imaginary part lead still to unphysical states, since the corresponding amplitude increase exponentially in a given channel, but such divergences at large distances are compensated by exponential decreasing amplitudes in another channel, giving an overall conservation of probability.
In particular, purely imaginary negative poles, that can be realized in our δ-potential case by considering $g < 0$, take the name of *virtual states*.
With $k_1 \neq 0$, instead, we have a so-called *resonance*, since it can be shown [9] that the corresponding cross section takes the typical shape of a Breit-Wigner distribution.
In summary, if we parametrize the S-matrix with the energy E, then $S(E)$ has a cut on the positive real axis and the region $\mathrm{Im}(k) > 0$ corresponds to the first (physical) sheet, while the region $\mathrm{Im}(k) < 0$ maps to the second or unphysical sheet. Moreover, poles on the negative real axis in the physical sheet correspond to bound states, resonances and virtual states are poles on the unphysical sheet, with the latter placed on the negative real axis, as in Figure 5.

5.2 Bound states and bootstrap equations

Close to a simple pole $\theta = \mathrm{i}u_{ij}^n$, corresponding to a bound state n formed by two particles i and j, a generic relativistic S-matrix can be written as (see Figure 6)

$$S_{ij}^{kl}(\theta) \simeq \frac{\Gamma_{ij}^n R_n \Gamma_n^{kl}}{\theta - \mathrm{i}u_{ij}^n}\,, \tag{5.11}$$

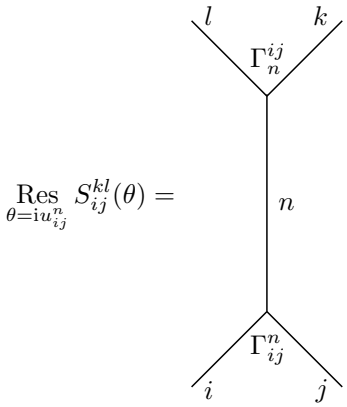

Figure 6: Scattering process associated to a bound state.

where R_n is the residue and $\Gamma_{ij}^n, \Gamma_n^{kl}$ are projectors of single particle (i, j, k and l) spaces onto the space of the bound state n.
The mass of the bound state is given by

$$s = m_n^2 = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^n \,. \tag{5.12}$$

It is interesting to notice that this relation has the geometrical meaning of the Carnot theorem for the triangle of the masses, as illustrated in Figure 7.
The main idea of the *bootstrap approach* is that the bound states can be considered on the same footing as the asymptotic states describing fundamental particles, even though the bound states can have bigger masses. Indeed, the ZF element describing bound states can be formally defined as

$$B_n(\theta) = \lim_{\epsilon\to 0} A_i(\theta - \mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}} - \epsilon) A_j(\theta + \mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}} + \epsilon) \,, \tag{5.13}$$

where $\bar{u} = \pi - u$ and the angles $u_{i\bar{n}}^{\bar{j}}$, $u_{j\bar{n}}^{\bar{i}}$ are defined according to the l.h.s. of Figure 8.
Therefore, the S-matrix for the scattering between any particle k and a bound state n, formed by the fusion of the particles i and j, can be derived by using the new bound state ZF elements (5.13): in a simple diagonal case it is given by the following product of fundamental diagonal S-matrices:

$$S_{kn}(\theta) = S_{ki}(\theta - \mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}}) S_{kj}(\theta + \mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}}) \,. \tag{5.14}$$

In the non-diagonal case, the S-matrix is projected onto the bound states channel by the vertex functions defined by (5.11) (see Figure 8):

$$\Gamma_{ij}^n S_{kn}^{ln'}(\theta) = S_{ki}^{k'i'}(\theta - \mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}}) S_{k'j}^{lj'}(\theta + \mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}}) \Gamma_{i'j'}^{n'} \,, \tag{5.15}$$

where the repeated indices are summed over $1, \ldots, N$, with N being the dimension of the symmetry algebra.
In this way we can take into account the possibility to have non-diagonal scattering between fundamental particles and bound states. This is the case of the $SU(3)$ chiral Gross-Neveu

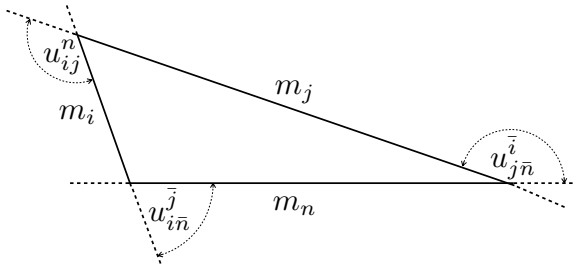

Figure 7: Carnot theorem interpretation of relation (5.12).

model, for instance, that will be discussed in Section 8.2. However, usually bound states and fundamental particles have different masses and then they scatter diagonally: this means that $k = l$ and $n = n'$ in (5.15), that reduces to

$$\Gamma_{ij}^{n} S_{kn}(\theta) = S_{ki}^{k'i'}(\theta - \mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}}) S_{k'j}^{kj'}(\theta + \mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}}) \Gamma_{i'j'}^{n} \,. \tag{5.16}$$

Furthermore, the bound state-bound state S-matrix can be calculated by

$$\Gamma_{ij}^{n} S_{mn}^{m'n'}(\theta) = S_{mi}^{lj'}(\theta - \mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}}) S_{lj}^{m'j'}(\theta + \mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}}) \Gamma_{n'}^{i'j'} \,, \tag{5.17}$$

namely by replacing the incoming (outgoing) particle k (l) in Figure 8 by the incoming (outgoing) bound state m (m'). In this way it is possible to compute all the S-matrices for all the bound-states of the theory. We will see in Sections 8.1.2 and 8.2.2 some concrete use of these equations to derive the corresponding bound states S-matrices.
In terms of the ZF algebra elements, we can rewrite (5.13) in a more formal way as

$$A_i(\theta_1) A_j(\theta_2) = \sum_n N_{ij}^{n} B_n \left(\frac{\theta_1 + \theta_2}{2} \right) \Bigg|_{\theta_1 - \theta_2 = \mathrm{i}\bar{u}_{ij}^{n}} \,, \tag{5.18}$$

where N_{ij}^{n} is 1 if B_n is a bound state of A_i and A_j and 0 otherwise. The *fusion rules* (5.18) must be consistent with the symmetries: $N_{ij}^{n} \neq 0$ only if charges C_i satisfy $C_n = C_i + C_j$. Then the bootstrap entails constraints on the charges. For example, given some charges eigenvalues with spin s $\omega_n^s(\theta) = \gamma_n^s e^{s\theta}$, then these have to satisfy the following *consistency bootstrap equations*

$$\gamma_n^s = \gamma_i^s e^{\mathrm{i}\bar{u}_{i\bar{n}}^{\bar{j}}} + \gamma_j^s e^{-\mathrm{i}\bar{u}_{j\bar{n}}^{\bar{i}}} \,. \tag{5.19}$$

6 Hopf algebra interpretation

The Hopf algebras (see part of [31] and [39] as introductory reviews on this subject) can be an useful tool for writing in a full algebraic way the symmetries of an S-matrix and to determine completely the S-matrix itself. The basic idea is to add to generic algebras some structures allowing the rigorous definition of operations over tensor products of representations, necessary to define multi-particle states with additive quantum numbers.

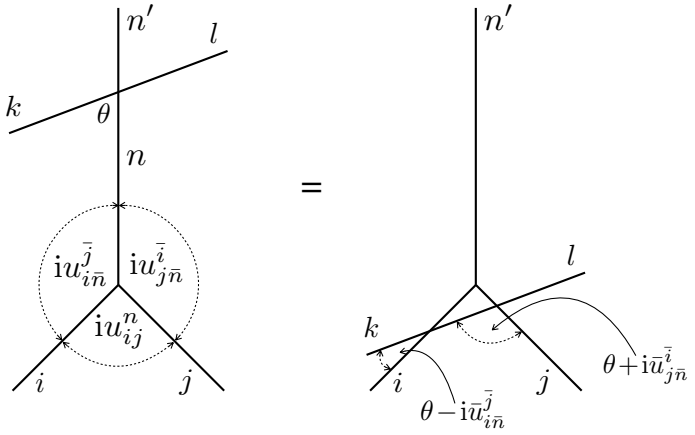

Figure 8: Bootstrap equation.

Let us consider, as an example, the universal enveloping of a Lie algebra. It is the tensor algebra $T(g)$ of a Lie algebra g: $\oplus_{n=0}^{\infty} g^{\otimes n}$. It has a multiplication corresponding to the tensor product

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m \,. \tag{6.1}$$

Then the quotient algebra $U(g) = T(g)/\mathbf{I}$, where $\mathbf{I}$ is the ideal generated by elements of the form $AB - BA - [A, B]$, with $A, B \in g$, is a Hopf algebra if a coproduct Δ, a counit ϵ and an antipode Σ are defined (see [31]). In particular, in this case they are explicitly given, $\forall J \in g$, by

$$\Delta(J) = J \otimes \mathbb{I} + \mathbb{I} \otimes J\,; \quad \Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}\,; \tag{6.2}$$

$$\epsilon(J) = 0\,; \quad \epsilon(\mathbb{I}) = \mathbb{I}\,; \tag{6.3}$$

$$\Sigma(J) = -J\,; \quad \Sigma(\mathbb{I}) = \mathbb{I}\,. \tag{6.4}$$

So, for example, if applied to the spin operator S_z in a space of 2-particle states classified by the spin eigenvalues s_1 and s_2, the coproduct gives

$$\Delta S_z |s_1 s_2\rangle = (S_z \otimes \mathbb{I} + \mathbb{I} \otimes S_z)|s_1 s_2\rangle = (s_1 + s_2)|s_1 s_2\rangle \,, \tag{6.5}$$

that is exactly what one expects from the action of a Lie algebra generator on a tensor product state. In order to generalize the action of Lie algebras on higher tensor products, higher coproducts can be defined as follows

$$\Delta^{(2)}(J) = (\mathbb{I} \otimes \Delta)\Delta(J) = (\Delta \otimes \mathbb{I})\Delta(J) = \mathbb{I} \otimes \mathbb{I} \otimes J + \mathbb{I} \otimes J \otimes \mathbb{I} + J \otimes \mathbb{I} \otimes \mathbb{I}\,, \tag{6.6}$$

$$\Delta^{(n)} = (\mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \Delta)\Delta^{(n-1)}\,, \tag{6.7}$$

still giving the desired action of the algebra as a sum of the actions on the single states involved in the tensor product state.

As we have already seen in Section 4.3, when we act with a symmetry generator J on a 2-particle state that belongs to a tensor product of two representations, we compute:

$$(J \otimes \mathbb{I} + \mathbb{I} \otimes J)|p_1, p_2\rangle \,. \tag{6.8}$$

Thus, the condition (4.57) for the compatibility of the S-matrix with a given symmetry algebra can be rewritten as

$$[\Delta(J), S] = 0 \,. \tag{6.9}$$

Moreover, if we equip the symmetry algebra with an antipode Σ, the antiparticle representation can be derived by

$$\pi\left[\Sigma(J)\right] = \mathcal{C}^{-1}\bar{\pi}(J)^t\mathcal{C} \,, \tag{6.10}$$

where $\mathcal{C}$ is the charge-conjugation matrix, π denotes the matrix representation and the superscript t means transposition.
Now, let us consider a *quasi cocommutative* Hopf algebra $\mathcal{A}$ (see [31] for the particular case of Yangians). By definition, this is equipped with an invertible element $\mathcal{R}$ belonging to $\mathcal{A} \otimes \mathcal{A}$ such that

$$\Delta^{op}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \,; \quad \forall a \in \mathcal{A} \,, \tag{6.11}$$

where $\Delta^{op} = P\Delta$, P is the permutation operator and $\mathcal{R}$ can be written as the sum $\mathcal{R} = \sum_{i,j} r_i \otimes r_j$, with $r_i \in \mathcal{A}$. Let us recall the properties satisfied by $\mathcal{R}$: in particular, if we define

$$\mathcal{R}_{12} = \mathcal{R} \otimes \mathbb{I} \,; \; \mathcal{R}_{23} = \mathbb{I} \otimes \mathcal{R} \,; \; \mathcal{R}_{13} = \sum_{ij} r_i \otimes \mathbb{I} \otimes r_j \,, \tag{6.12}$$

a quasi commutative Hopf algebra is called *quasi triangular* if

$$(\Delta \otimes \mathbb{I})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \,, \tag{6.13}$$

$$(\mathbb{I} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \,, \tag{6.14}$$

and $\mathcal{R}$ is called *universal R-matrix*.
It can be shown [40] that the universal R-matrix of a quasi triangular Hopf algebra satisfies

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \,, \tag{6.15}$$

$$(\Sigma \otimes \mathbb{I})\mathcal{R} = (\mathbb{I} \otimes \Sigma^{-1})\mathcal{R} = \mathcal{R}^{-1} \,. \tag{6.16}$$

Relation (6.15) is obtained by comparing the expression of $(\mathbb{I} \otimes \Delta^{op})\mathcal{R}$ written as

$$(\mathbb{I} \otimes P\Delta)\mathcal{R} = (\mathbb{I} \otimes P)(\mathbb{I} \otimes \Delta)\mathcal{R} = (\mathbb{I} \otimes P)\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}_{13} \,, \tag{6.17}$$

where we used (6.14), and

$$(\mathbb{I} \otimes \Delta^{op})\mathcal{R} = (\mathbb{I} \otimes \mathcal{R}\Delta\mathcal{R}^{-1})\mathcal{R} = (\mathbb{I} \otimes \mathcal{R})(\mathbb{I} \otimes \Delta)\mathcal{R}(\mathbb{I} \otimes \mathcal{R}^{-1}) = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1} \,, \tag{6.18}$$

where definitions (6.11) and (6.12) have been used. Thus, the comparison of (6.18) with (6.17) gives (6.15). For a demonstration of (6.16), the interested reader can look at Section 2.2.1 of [43], for instance.

A spectral parameter θ can be introduced by an automorphism D_θ of the Hopf algebra $\mathcal{A}$, such that $D_\theta D_{\theta'} = D_{\theta+\theta'}$, $D_0 = 1$ and

$$(\mathbb{I} \otimes D_\theta)\mathcal{R} = (D_{-\theta} \otimes \mathbb{I})\mathcal{R} = \mathcal{R}(\theta)\,. \tag{6.19}$$

Then (6.15) becomes

$$\mathcal{R}_{12}(\theta)\mathcal{R}_{13}(\theta+\theta')\mathcal{R}_{23}(\theta') = \mathcal{R}_{23}(\theta')\mathcal{R}_{13}(\theta+\theta')\mathcal{R}_{12}(\theta)\,, \tag{6.20}$$

and its matrix representation, with the identification $S = P\mathcal{R}$, gives the YBE (3.11).
It can be also shown that properties (6.16) and (6.13)-(6.14) are respectively equivalent to the *crossing symmetry* [41] and the *bootstrap equations* (5.14) for the S-matrix [42]. Therefore, this algebraic formulation, alternative to the one mentioned in Section 4.2, can be useful to introduce the concept of crossing symmetry in non-relativistic theories, as done in [17] for the AdS_5/CFT_4 case, for instance.

7 Form factors

The knowledge of the 2-particle S-matrix in an integrable theory is a fundamental step towards the determination of its correlation functions, that are necessary to calculate the physical quantities of the model.
Indeed, an essential ingredient for the full solution of a (1+1)-dimensional integrable theory is the determination of its generalized form factors[6], that are the matrix elements of local operators evaluated between *out* and *in* asymptotic states:

$${}^{out}_{b_1\dots b_m}\langle\theta'_1\dots\theta'_m|\mathcal{O}(x)|\theta_{m+1}\dots\theta_n\rangle^{in}_{a_{m+1}\dots a_n}\,. \tag{7.1}$$

We will see how these are deeply related to the S-matrix and the bootstrap program discussed in the previous Sections.
The correlation functions can be related to a special class of generalized form factors by inserting[7]

$$1 = \sum_{n=0}^{\infty}\int \frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}|\theta_1,\dots,\theta_n\rangle^{in}_{\underline{a}}\,{}^{in}_{\underline{a}}\langle\theta_1,\dots,\theta_n| \tag{7.2}$$

into a two-point function

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty}\int \frac{d\theta_1\dots d\theta_n}{n!(2\pi)^n}\langle 0|\mathcal{O}(x)|\theta_1,\dots,\theta_n\rangle^{in}_{\underline{a}}\,{}^{in}_{\underline{a}}\langle\theta_1,\dots,\theta_n|\mathcal{O}(0)|0\rangle\,. \tag{7.3}$$

We see indeed that this involves the actual *form factor*

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1,\dots,\theta_n) = \langle 0|\mathcal{O}(0)|\theta_1,\dots,\theta_n\rangle^{in}_{\underline{a}}\,, \tag{7.4}$$

that is indeed defined as the matrix element of a local operator placed at the origin, between an n-particle state and the vacuum.

[6] Though the form factor program succeeded in calculating exactly the correlation functions of the Ising model [44] only.

[7] In what follows we will collect the color labels $a_1,\dots,a_n$ in the notation $\underline{a}$.

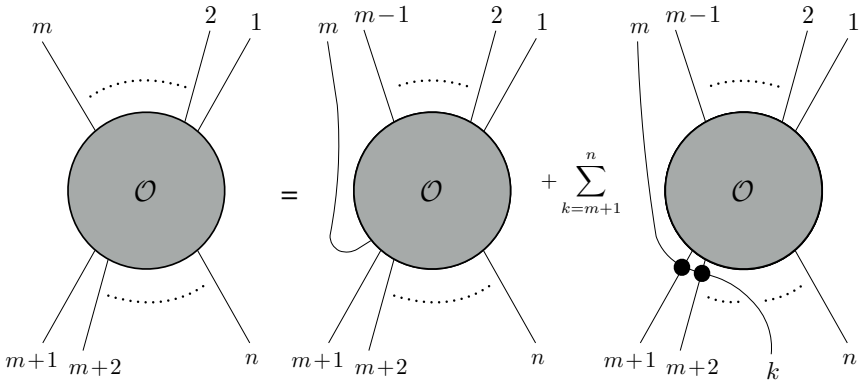

Figure 9: Crossing relation for the form factors.

As for the S-matrix, let us discuss the properties satisfied by the form factors $F^{\mathcal{O}}_{\underline{a}}(\underline{\theta})$. From the constraints given by these properties we will get fundamental hints to find their general solutions.

First, in the case of local scalar operators $\mathcal{O}(x)$, *relativistic invariance* implies that the form factors are functions of the rapidities differences $\theta_{ij} = \theta_i - \theta_j$:

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) = F^{\mathcal{O}}_{\underline{a}}(\theta_{12}, \theta_{13}, \dots, \theta_{ij}, \dots, \theta_{n-1n}) \; ; \quad i < j \, . \tag{7.5}$$

For operators of generic spin s, we have instead

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1 + \Lambda, \dots, \theta_n + \Lambda) = e^{s\Lambda} F^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) \, . \tag{7.6}$$

In what follows we will focus on the case of scalar operators.

It is possible to show that *CPT invariance* implies, under replacement of *in* by *out* states, the following simple relation

$$\langle 0|\mathcal{O}(0)|\theta_1, \dots, \theta_n\rangle^{out}_{\underline{a}} = F^{\mathcal{O}}_{\underline{a}}(-\theta_{ij}) \; ; \quad 1 \le i < j \le n \, . \tag{7.7}$$

The general property satisfied when a particle is moved from the *out* to the *in* state, instead, takes the name of *crossing*: it is depicted in Figure 9 and is formalized by the following relation[8] (see [45] for instance):

$$\begin{aligned} &{}^{out}_{i_1 \dots i_m}\langle \theta'_1, \dots, \theta'_m|\mathcal{O}(0)|\theta_{m+1}, \dots, \theta_n\rangle^{in}_{j_{m+1} \dots j_n} \equiv F^{\mathcal{O}}_{i_1 \dots i_m; j_{m+1} \dots j_n}(\theta'_1, \dots, \theta'_m|\theta_{m+1}, \dots, \theta_n) \\ &= F^{\mathcal{O}}_{i_1 \dots i_{m-1}; i_m j_{m+1} \dots j_n}(\theta'_1, \dots, \theta'_{m-1}|\theta'_m + \mathrm{i}\pi, \theta_{m+1}, \dots, \theta_n) + \sum_{k=m+1}^{n} \delta_{i_m j_k} \delta(\theta'_m - \theta_k) \\ &\times \prod_{l=1}^{k-1} S_{j_l j_k}(\theta_l - \theta_k) F_{i_1 \dots i_{m-1}; j_{m+1} \dots j_{k-1} j_{k+1} \dots j_n}(\theta'_1, \dots, \theta'_{m-1}|\theta_{m+1}, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n) \, . \end{aligned} \tag{7.8}$$

For example, in the 2-particle case, this property reads

$${}^{out}_{a_1}\langle \theta_1|\mathcal{O}(0)|\theta_2\rangle^{in}_{a_2} = F^{\mathcal{O}}_{a_1 a_2}(\theta_{12} + \mathrm{i}\pi) + \delta_{a_1 a_2} \delta(\theta_{12}) \langle \mathcal{O}\rangle \, . \tag{7.9}$$

[8] All the formulas of this Section, for simplicity, are written for a *diagonal* case with neutral particles.

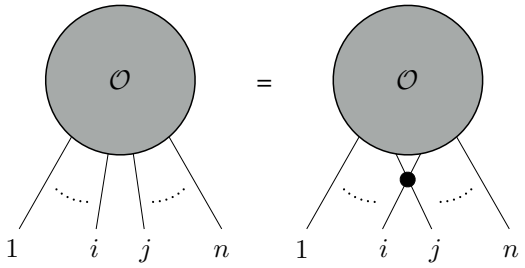

Figure 10: Watson equation for permutation of two particles.

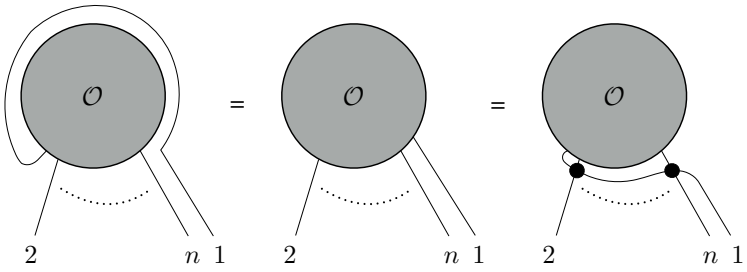

Figure 11: Watson equation for periodicity under shifts of $2\pi\mathrm{i}$.

Here we just stated formulas (7.8) and (7.7) without proof, however it is possible to derive them on the basis of the the LSZ reduction formalism [46] and the maximal analyticity assumption, *i.e.* possible singularities of the form factors can occur only due to physical processes like the formation of bound states, similarly to the analyticity property assumed for the S-matrix. The related derivations can be found in Appendix A of [24], for instance. The symmetry properties satisfied under permutations of θ_i, θ_j and shifts by $2\pi\mathrm{i}$, represented in Figures 10 and 11 respectively, are called *Watson equations* after [47], and in a diagonal case they read

$$\begin{aligned} &F^{\mathcal{O}}_{a_1\dots a_i a_j\dots a_n}(\theta_1,\dots,\theta_i,\theta_j,\dots,\theta_n) \\ &\qquad = F^{\mathcal{O}}_{a_1\dots a_j a_i\dots a_n}(\theta_1,\dots,\theta_j,\theta_i,\dots,\theta_n) S_{a_i a_j}(\theta_{ij}) \;; \quad j = i+1\,, \end{aligned} \tag{7.10}$$

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1 + 2\pi\mathrm{i},\dots,\theta_n) = F^{\mathcal{O}}_{a_2\dots a_n a_1}(\theta_2,\dots,\theta_n,\theta_1) = \prod_{i=2}^{n} S_{a_i a_1}(\theta_i - \theta_1) F^{\mathcal{O}}_{\underline{a}}(\theta_1,\dots,\theta_n)\,. \tag{7.11}$$

They can be derived, in the case $n = 2$ for example, by using the definition of the S-matrix, factorization and CPT invariance:

$$\begin{aligned} F^{\mathcal{O}}_{a_1 a_2}(\theta_{12}) &= \langle 0|\mathcal{O}(0)|\theta_1\theta_2\rangle^{in}_{a_1 a_2} = \langle 0|\mathcal{O}(0)|\theta_1\theta_2\rangle^{out}_{a_1 a_2} S_{a_1 a_2}(\theta_{12}) \\ &= F^{\mathcal{O}}_{a_2 a_1}(-\theta_{12}) S_{a_1 a_2}(\theta_{12})\,, \end{aligned} \tag{7.12}$$

$$\begin{aligned} F^{\mathcal{O}}_{a_2 a_1}(\mathrm{i}\pi - \theta_{12}) &= {}^{out}_{a_1}\langle\theta_1|\mathcal{O}(0)|\theta_2\rangle^{in}_{a_2} = {}^{in}_{a_1}\langle\theta_1|\mathcal{O}(0)|\theta_2\rangle^{out}_{a_2} \\ &= F^{\mathcal{O}}_{a_1 a_2}(\mathrm{i}\pi + \theta_{12})\,, \end{aligned} \tag{7.13}$$

where the next-to-last identity is due to the triviality of the 1-particle S-matrix.

As in the case of the S-matrix, we look for general solutions of the Watson equations and the other conditions listed above in the form of a minimal solution $F^{min}_{\underline{a}}(\theta_{ij})$, without poles and zeros in the physical strip $0 \leq \mathrm{Im}(\theta_{ij}) \leq \pi$, multiplied by a factor $K_{\underline{a}}(\theta_{ij})$ containing all the information about the poles (zeros) structure. For scalar operators, this reads

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) = K^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) \prod_{i<j} F^{min}_{a_i a_j}(\theta_{ij}) \,. \tag{7.14}$$

In the case of $n = 2$, we are saying that the most general solution of the Watson equations [21]

$$F^{\mathcal{O}}_{a_1 a_2}(\theta_{12}) = F^{\mathcal{O}}_{a_2 a_1}(-\theta_{12}) S_{a_1 a_2}(\theta_{12}) \,, \quad F^{\mathcal{O}}_{a_1 a_2}(\mathrm{i}\pi - \theta_{12}) = F^{\mathcal{O}}_{a_2 a_1}(\mathrm{i}\pi + \theta_{12}) \,, \tag{7.15}$$

is given by $F^{\mathcal{O}}_{a_1 a_2}(\theta) = K^{\mathcal{O}}_{a_1 a_2}(\theta) F^{min}_{a_1 a_2}(\theta)$, with $K^{\mathcal{O}}_{a_1 a_2}(\theta)$ satisfying

$$K^{\mathcal{O}}_{a_1 a_2}(\theta) = K^{\mathcal{O}}_{a_2 a_1}(-\theta) = K^{\mathcal{O}}_{a_1 a_2}(2\pi\mathrm{i} + \theta) \,. \tag{7.16}$$

If $\pm i\alpha_1, \dots, \pm \mathrm{i}\alpha_L$ are poles of $F^{\mathcal{O}}_{a_1 a_2}(\theta)$ in the physical strip, then

$$K^{\mathcal{O}}_{a_1 a_2}(\theta) = N^{\mathcal{O}}(\theta) \prod_{k=1}^{L} \frac{1}{\sinh \frac{\theta - \mathrm{i}\alpha_k}{2} \sinh \frac{\theta + \mathrm{i}\alpha_k}{2}} \,. \tag{7.17}$$

For scalar operators, as we will see in the examples of Sections 8, the normalization factor $N^{\mathcal{O}}(\theta)$ is a constant and the poles of $K^{\mathcal{O}}(\theta)$ contain all the information about the operator $\mathcal{O}$. If in addition $\langle \mathcal{O} \rangle = 0$, $N^{\mathcal{O}}$ can be fixed using relation (7.9):

$${}_a\langle \theta | \mathcal{O} | \theta \rangle_a = F^{\mathcal{O}}_{aa}(\mathrm{i}\pi) \,. \tag{7.18}$$

On the other hand, Cauchy theorem implies that, given a contour C enclosing the strip $0 \leq \mathrm{Im}(\theta) \leq 2\pi$, $F^{min}_{a_1 a_2}(\theta)$ satisfies

$$\begin{aligned} \frac{d}{d\theta} \log F^{min}_{a_1 a_2}(\theta) &= \frac{1}{8\pi\mathrm{i}} \int_C \frac{dz}{\sinh^2 \frac{z-\theta}{2}} \log F^{min}_{a_1 a_2}(z) \\ &= \frac{1}{8\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{dz}{\sinh^2 \frac{z-\theta}{2}} \log \frac{F^{min}_{a_1 a_2}(z)}{F^{min}_{a_1 a_2}(z + 2\pi\mathrm{i})} = \frac{1}{8\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{dz}{\sinh^2 \frac{z-\theta}{2}} \log S_{a_1 a_2}(z) \,, \end{aligned} \tag{7.19}$$

where we used the property (7.11) in the last equality. Then we can calculate the minimal solution $F^{min}_{a_1 a_2}(\theta)$ entirely from the S-matrix element $S_{a_1 a_2}(\theta)$!
Regarding the factor $K^{\mathcal{O}}(\theta_{12})$, it has to satisfy the Watson equations with $S(\theta_{12}) = 1$, then it is symmetric in θ_{12} and periodic with period $2\pi\mathrm{i}$, *i.e.* it is function of $\cosh \theta_{12}$.
In general, n-particle functions $K^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n)$ have poles when a cluster of k particles have the kinematic configuration of a 1-particle state. In particular, this happens when the set of *in* particles contains a particle-antiparticle pair with opposite momenta, *e.g.* $\theta_{12} = \mathrm{i}\pi$ (see Figure 12):

$$\operatorname*{Res}_{\theta_{12} = \mathrm{i}\pi} F^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_i, \theta_j, \dots, \theta_n) = 2\mathrm{i} F^{\mathcal{O}}_{a_3 \dots a_n}(\theta_3, \dots, \theta_n) \left[1 - \prod_{i=3}^{n} S_{a_2 a_i}(\theta_2 - \theta_i) \right] , \tag{7.20}$$

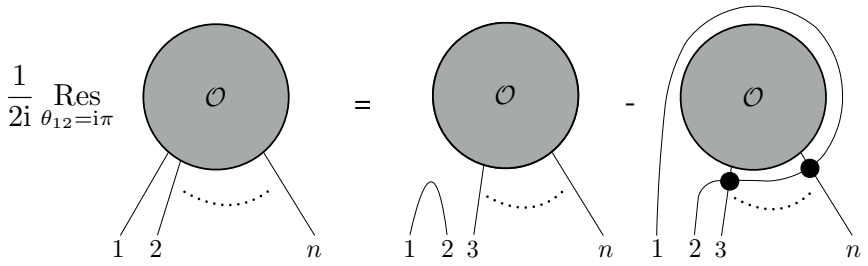

Figure 12: Recursive relation from the residue at $\theta_{12} = \mathrm{i}\pi$.

that gives a recursive relation between n- and $n-2$-particle form factors. Property (7.20) follows from realizing that in (7.8) the particle m can be also moved to end of the *in* particles set:

$$F^{\mathcal{O}}_{i_1\dots i_m;j_{m+1}\dots j_n}(\theta'_1,\dots,\theta'_m|\theta_{m+1},\dots,\theta_n) \tag{7.21}$$
$$= F^{\mathcal{O}}_{i_1\dots i_{m-1};j_{m+1}\dots j_n i_m}(\theta'_1,\dots,\theta'_{m-1}|\theta_{m+1},\dots,\theta_n,\theta'_m-\mathrm{i}\pi) + \sum_{k=m+1}^{n}\delta_{i_m j_k}\delta(\theta'_m-\theta_k)$$
$$\times\prod_{l=k+1}^{n} S_{j_l j_k}(\theta_l-\theta_k)F^{\mathcal{O}}_{i_1\dots i_{m-1};j_{m+1}\dots j_{k-1}j_{k+1}\dots j_n}(\theta'_1,\dots,\theta'_{m-1}|\theta_{m+1},\dots,\theta_{k-1},\theta_{k+1},\dots,\theta_n)\,.$$

Thus, comparing the analytic parts of the crossing relations (7.8) and (7.21) we can obtain the first periodicity relation in (7.11), and if we evaluate that at $\theta_1 \sim \theta_2$ we get

$$F^{\mathcal{O}}_{a_1 a_2\dots a_n}(\theta_1+\mathrm{i}\pi,\theta_2,\dots,\theta_n) \sim \frac{f(\theta_2,\dots,\theta_n)}{\theta_1-\theta_2-\mathrm{i}\epsilon}\,, \tag{7.22}$$
$$F^{\mathcal{O}}_{a_2\dots a_n a_1}(\theta_2,\dots,\theta_n,\theta_1-\mathrm{i}\pi) \sim \frac{f(\theta_2,\dots,\theta_n)}{\theta_1-\theta_2+\mathrm{i}\epsilon}\,, \tag{7.23}$$

for some function f and small ϵ. Hence, plugging (7.22) and (7.23) into (7.8) and (7.21) respectively, in the case $m=1$ and evaluated at $\theta_1 \sim \theta_2$, and comparing the δ-function parts, one gets

$$f(\theta_2,\dots,\theta_n) = 2\mathrm{i}F^{\mathcal{O}}_{a_3\dots a_n}(\theta_3,\dots,\theta_n)\left[1-\prod_{i=3}^{n}S_{a_2 a_i}(\theta_2-\theta_i)\right], \tag{7.24}$$

and then (7.20).

A further recursive relation, depicted in Figure 13, connects n- and $n-1$-particle form factors if there is a bound state pole at $\theta_{12} = \mathrm{i}u_{12}^{(12)}$, for example[9]:

$$\operatorname*{Res}_{\theta_{12}=\mathrm{i}u_{12}^{(12)}} F^{\mathcal{O}}_{\underline{a}}(\theta_1,\dots,\theta_n) = \sqrt{2\mathrm{i}R_{(12)}}\Gamma_{12}^{(12)}F^{\mathcal{O}}_{a_{(12)}a_3\dots a_n}(\theta_{(12)},\theta_3\dots,\theta_n)\,, \tag{7.25}$$

[9]By $\mathrm{i}u_{12}^{(12)}$, we denote the position of the pole corresponding to the bound state (12) made of particles 1 and 2.

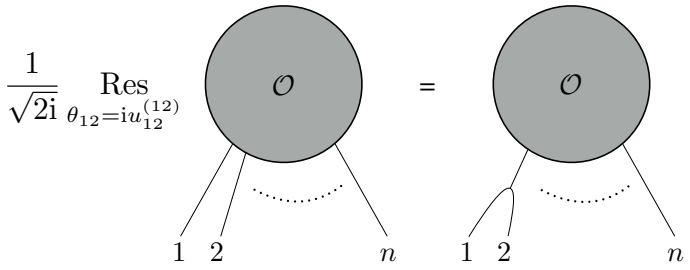

Figure 13: Recursive relation from the residue at $\theta_{12} = \mathrm{i}u_{12}^{(12)}$.

where $\theta_{(12)} = (\theta_1 + \theta_2)/2$, $R_{(12)}$ is the residue of the S-matrix at $\theta = \mathrm{i}u_{12}^{(12)}$ and $\Gamma_{12}^{(12)}$ projects the spaces of particles 1 and 2 onto the space of the bound state (12), as defined in (5.11).
A derivation of (7.25), making use of two-point correlators, the Watson equation (7.10) and the residue of the S-matrix (5.11), can be found, as all the others discussed in this Section, in Appendix A of [24]. A few examples of solutions in very simple cases are given in Sections 8.1.3 and 8.2.3. The interested reader can look at [9,21,22,24,25,72] for further details.

8 Examples

As promised, in this Section we specialize the properties and results of S-matrices and form factors, discussed above for generic (1+1)-dimensional integrable theories, to two relevant examples of quantum integrable relativistic models: sine-Gordon and chiral Gross-Neveu. At the end of the Section, we will also summarize recent developments about the S-matrices of AdS/CFT correspondences.

8.1 Sine-Gordon

The quantum sine-Gordon model (see [73] for the discussion of the classical theory) is a (1+1)-dimensional integrable[10] theory of a bosonic scalar field ϕ, described by the following Lagrangian density:

$$\mathcal{L}_{sG} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{\beta^2}(\cos\beta\phi - 1)\,, \tag{8.1}$$

where $\mu = 0, 1$ and β is a coupling constant. In what follows we will use a parameter ξ given by

$$\xi = \frac{\beta^2}{8}\frac{1}{1-\frac{\beta^2}{8\pi}}\,. \tag{8.2}$$

In particular, the coupling constant defines two distinct regions for $\beta^2 < 4\pi$ ($\xi < \pi$) and $\beta^2 > 4\pi$ ($\xi > \pi$), which are called respectively *attractive* and *repulsive* regimes.

[10] Its quantum integrability has been shown in [48, 49].

These names are due to the presence of bound states solutions in the attractive case and their absence in the repulsive one. As we will see at the end of this Section, the elementary excitation of the bosonic field ϕ corresponds to the bound state of a soliton and an antisoliton, that classically are solutions of the equation of motion associated to the Lagrangian (8.1), reviewed in [73]. The quantized solitons are the fundamental excitations interacting through the S-matrix that we are going to study in the next Section. As shown in [50, 51], they can be put in correspondence with the self-interacting Dirac fermions described by a (1+1)-dimensional theory, called massive Thirring model (MTM), defined by the following Lagrangian density

$$\mathcal{L}_{MTM} = \bar{\psi}(\mathrm{i}\gamma_\mu \partial^\mu - m)\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi\,, \tag{8.3}$$

where γ^μ are the two-dimensional Dirac matrices and g is a coupling constant related to the sine-Gordon β as [50]

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}\,. \tag{8.4}$$

In particular, at $\beta^2 = 4\pi$ ($\xi = \pi$) the theory describes a free fermion.
The sine-Gordon model possesses an $O(2)$ symmetry, and we will use this to constrain the matrix form of the S-matrix. In general, the $O(N)$ symmetry tells us that the spectrum of the fundamental excitations consists of a multiplet of N particles of equal mass, denoted by $A_i\,, i = 1, \ldots, N$. Moreover, the commutation relations corresponding to (4.25) are constrained to be [11]

$$A_i(\theta_1)A_j(\theta_2) = \delta_{ij}S_1(\theta_1 - \theta_2)\sum_{k=1}^{N} A_k(\theta_2)A_k(\theta_1) \tag{8.5}$$

$$+S_2(\theta_1 - \theta_2)A_j(\theta_2)A_i(\theta_1) + S_3(\theta_1 - \theta_2)A_i(\theta_2)A_j(\theta_1)\,. \tag{8.6}$$

8.1.1 Solution for the exact S-matrix

The $O(2)$ symmetry group is the group of orthogonal matrices in two dimensions, and its Lie algebra is generated by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,. \tag{8.7}$$

Now, we can equip this algebra with the operations and properties of a Hopf algebra and in particular we can impose the invariance of the S-matrix under $O(2)$ by using the coproduct (6.2)

$$[\Delta(J), S] = 0\,. \tag{8.8}$$

Solving the system of equations given by (8.8) and requiring parity and time reversal invariances, one gets the following matrix structure:

$$S_{sG} = \begin{pmatrix} S_1 + S_2 + S_3 & & & S_1 \\ & S_2 & S_3 & \\ & S_3 & S_2 & \\ S_1 & & & S_1 + S_2 + S_3 \end{pmatrix}\,. \tag{8.9}$$

Written in terms of the ZF elements $A_1(\theta), A_2(\theta)$, this is equivalent to (8.6) for $N = 2$. Following [11], we define the soliton and antisoliton ZF elements as

$$\text{soliton} \qquad s(\theta) = A_1(\theta) + \mathrm{i}A_2(\theta)\,, \tag{8.10}$$

$$\text{antisoliton} \quad \bar{s}(\theta) = A_1(\theta) - \mathrm{i}A_2(\theta)\,. \tag{8.11}$$

In this new basis, the ZF commutation relations become

$$s(\theta_1)\bar{s}(\theta_2) = S_T(\theta_1-\theta_2)\bar{s}(\theta_2)s(\theta_1) + S_R(\theta_1-\theta_2)s(\theta_2)\bar{s}(\theta_1)\,, \tag{8.12}$$

$$s(\theta_1)s(\theta_2) = S(\theta_1-\theta_2)s(\theta_2)s(\theta_1)\,, \tag{8.13}$$

$$\bar{s}(\theta_1)\bar{s}(\theta_2) = S(\theta_1-\theta_2)\bar{s}(\theta_2)\bar{s}(\theta_1)\,, \tag{8.14}$$

where S_T and S_R denote the transmission and reflection amplitudes respectively, and in terms of S_1, S_2 and S_3 they read

$$S(\theta) = S_3(\theta) + S_2(\theta)\,, \tag{8.15}$$

$$S_T(\theta) = S_1(\theta) + S_2(\theta)\,, \tag{8.16}$$

$$S_R(\theta) = S_1(\theta) + S_3(\theta)\,. \tag{8.17}$$

Then the S-matrix takes the form

$$S_{sG} = \begin{pmatrix} S & & & \\ & S_T & S_R & \\ & S_R & S_T & \\ & & & S \end{pmatrix}. \tag{8.18}$$

Imposing *crossing symmetry* on this S-matrix and using the charge conjugation matrix $\mathcal{C}_{ij} = \delta_{\bar{i}j}$, one gets

$$S(\theta) = S_T(\mathrm{i}\pi - \theta)\,, \quad S_R(\theta) = S_R(\mathrm{i}\pi - \theta)\,, \tag{8.19}$$

while *unitarity* entails

$$S(\theta)S(-\theta) = 1\,, \tag{8.20}$$

$$S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta) = 1\,, \tag{8.21}$$

$$S_T(\theta)S_R(-\theta) + S_R(\theta)S_T(-\theta) = 0\,. \tag{8.22}$$

The YBE (3.12) fixes the ratio S_T/S_R, as mentioned in Section 4.3. In details, imposing the condition (3.12), one obtains

$$S_R(\theta_{12})S_R(\theta_{13})S_T(\theta_{23}) + S_T(\theta_{12})S(\theta_{13})S_R(\theta_{23}) - S(\theta_{12})S_T(\theta_{13})S_R(\theta_{23}) = 0\,, \tag{8.23}$$

$$S_R(\theta_{12})S(\theta_{13})S_R(\theta_{23}) + S_T(\theta_{12})S_R(\theta_{13})S_T(\theta_{23}) - S(\theta_{12})S_R(\theta_{13})S(\theta_{23}) = 0\,. \tag{8.24}$$

These constraints were solved, in terms of the ratios S_2/S_3 and S_1/S_3 of the elements appearing in (8.9), in the Appendix A of [11]. In particular, those ratios were respectively given as solutions of differential equations obtained by differentiating the YBE (8.24), with boundary conditions satisfying crossing (the second of (8.19)) and unitarity (8.21)-(8.22).

Fulfilling all these constraints leaves actually a free parameter, that can be fixed to be proportional to ξ by comparison with semiclassical results [11]. Finally, S_R and S_T result to depend on S in the following way

$$S_T(\theta) = \frac{\sinh\frac{\pi\theta}{\xi}}{\sinh\frac{\pi(\mathrm{i}\pi-\theta)}{\xi}} S(\theta)\,, \quad S_R(\theta) = \frac{\mathrm{i}\sin\frac{\pi^2}{\xi}}{\sinh\frac{\pi(\mathrm{i}\pi-\theta)}{\xi}} S(\theta)\,. \tag{8.25}$$

Hence the crossing relation for $S(\theta)$ can be written as

$$S(\theta) = \frac{\sinh\frac{\pi(\mathrm{i}\pi-\theta)}{\xi}}{\sinh\frac{\pi\theta}{\xi}} S(\mathrm{i}\pi-\theta)\,. \tag{8.26}$$

Now, the first step to find a minimal solution of (8.20) and (8.26) for $S(\theta)$ is to write (8.26) in terms of Γ functions by using the property $\sinh\pi x = \pi\left[\Gamma(1+\mathrm{i}x)\Gamma(-\mathrm{i}x)\right]^{-1}$:

$$S(\theta) = \frac{\Gamma\left(1+\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left(-\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1-\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left(\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)} S(\mathrm{i}\pi-\theta)\,. \tag{8.27}$$

Then, taking an ansatz for $S(\theta)$ satisfying (8.27)

$$S(\theta) = \frac{\Gamma\left(1+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}\,, \tag{8.28}$$

we multiply it by a factor $f(\theta)$ such that the corrected $S(\theta)$ now satisfies unitarity (8.20)

$$f(\theta)f(-\theta)\frac{\Gamma\left(1+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}\frac{\Gamma\left(1-\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)} = 1 \;\Rightarrow\; f(\theta) = \frac{\Gamma\left(\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1-\mathrm{i}\frac{\theta}{\xi}\right)} g(\theta)\,, \tag{8.29}$$

with $g(\theta)$ such that $S(\theta)$ satisfies crossing again

$$\frac{\Gamma\left(\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1-\mathrm{i}\frac{\theta}{\xi}\right)} g(\theta) = \frac{\Gamma\left(\frac{2\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1+\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)} g(\mathrm{i}\pi-\theta) \;\Rightarrow\; g(\theta) = \frac{\Gamma\left(\frac{2\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1+\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)} h(\theta)\,, \tag{8.30}$$

and so on. At the end of this recursive procedure, one gets the infinite product

$$\begin{aligned} S(\theta) \;&=\; -\prod_{k=0}^{\infty} \frac{\Gamma\left(1+(2k+1)\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left(1+2k\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left(1+(2k+1)\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left(1+2k\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)} \\ &\times\; \frac{\Gamma\left((2k+1)\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left((2k+2)\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)}{\Gamma\left((2k+1)\frac{\pi}{\xi}+\mathrm{i}\frac{\theta}{\xi}\right)\Gamma\left((2k+2)\frac{\pi}{\xi}-\mathrm{i}\frac{\theta}{\xi}\right)}\,, \end{aligned} \tag{8.31}$$

where we put an overall minus sign since the sine-Gordon S-matrix, from the discussion in [52], has to satisfy $S_{aa}^{aa}(0) = -1$. The result (8.31) can be also derived using the

technique explained in [37,14]: introducing the shift operator $D \equiv e^{\frac{i\pi}{2}\partial_\theta}$, such that $Df(\theta) = f(\theta + i\pi/2)$ and $f^D = e^{D \log f}$, we can write the crossing relation (8.27) as

$$S(\theta)^{D+D^{-1}} = \frac{\left[\Gamma\left(1+i\frac{\theta}{\xi}\right)\Gamma\left(-i\frac{\theta}{\xi}\right)\right]^D}{\left[\Gamma\left(1-i\frac{\theta}{\xi}\right)\Gamma\left(i\frac{\theta}{\xi}\right)\right]^{D^{-1}}}\,, \tag{8.32}$$

that is formally solved by

$$S(\theta) = \frac{\left[\Gamma\left(1+i\frac{\theta}{\xi}\right)\Gamma\left(-i\frac{\theta}{\xi}\right)\right]^{\frac{D}{D+D^{-1}}}}{\left[\Gamma\left(1-i\frac{\theta}{\xi}\right)\Gamma\left(i\frac{\theta}{\xi}\right)\right]^{\frac{D^{-1}}{D+D^{-1}}}}\,. \tag{8.33}$$

The exponents $D^{\pm 1}/(D+D^{-1})$ can be expanded at small or at large D: the choice should be consistent with the minimality condition, *i.e.* the absence of zeros and poles in the physical strip, of the resulting $S(\theta)$. In particular, the factors in the r.h.s. of (8.33) can be written as

$$\Gamma\left(1+i\frac{\theta}{\xi}\right)^{\frac{D}{D+D^{-1}}} = \exp\left[-\sum_{n=1}^{\infty}(-1)^n D^{-2n+2}\log\Gamma\left(1+i\frac{\theta}{\xi}\right)\right]\,, \tag{8.34}$$

$$\Gamma\left(-i\frac{\theta}{\xi}\right)^{\frac{D}{D+D^{-1}}} = \exp\left[-\sum_{n=1}^{\infty}(-1)^n D^{2n}\log\Gamma\left(-i\frac{\theta}{\xi}\right)\right]\,, \tag{8.35}$$

$$\Gamma\left(1-i\frac{\theta}{\xi}\right)^{\frac{D^{-1}}{D+D^{-1}}} = \exp\left[-\sum_{n=1}^{\infty}(-1)^n D^{2n-2}\log\Gamma\left(1-i\frac{\theta}{\xi}\right)\right]\,, \tag{8.36}$$

$$\Gamma\left(i\frac{\theta}{\xi}\right)^{\frac{D^{-1}}{D+D^{-1}}} = \exp\left[-\sum_{n=1}^{\infty}(-1)^n D^{-2n}\log\Gamma\left(-i\frac{\theta}{\xi}\right)\right]\,. \tag{8.37}$$

Thus, we get the product (8.31). Regularizing the sums in the exponents introduces an overall constant, that is set to -1 by the aforementioned condition $S_{aa}^{aa}(0) = -1$. Moreover, using the following integral representation of $\log\Gamma$

$$\log\Gamma(x) = \int_0^\infty \frac{dt}{t}\left[(x-1)e^{-t} + \frac{e^{-tx}-e^{-t}}{1-e^{-t}}\right]\,, \tag{8.38}$$

(8.31) can be recast in the following compact integral form

$$S(\theta) = -\exp\left[-i\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{t(\pi-\xi)}{2}}{\sinh\frac{\xi t}{2}\cosh\frac{\pi t}{2}}\sin\theta t\right]\,. \tag{8.39}$$

At the specific value $\xi = \pi/N$, the soliton-soliton amplitude was already determined in [53, 54], later confirmed by the exact derivation, on the basis of crossing and unitarity, of [55]. In the limit of $\xi \to 0$, expressions (8.25) and (8.31) agree with the semiclassical results of [53, 56].

Another way to determine (8.39), that can be found in [57, 58], uses a trick similar to that used for the derivation of the 2-particle minimal form factor (7.19):

$$\log S_T(\theta) = \int_C \frac{dz}{2\pi \mathrm{i}} \frac{\log S_T(z)}{\sinh(z-\theta)} = \int_{-\infty}^{\infty} \frac{dz}{2\pi \mathrm{i}} \frac{\log S_T(z) S_T(\mathrm{i}\pi + z)}{\sinh(z-\theta)}\,, \tag{8.40}$$

where C is a contour encircling the strip $0 < \mathrm{Im}(\theta) < \pi$. Unitarity and crossing relations imply

$$S_T(z) S_T(\mathrm{i}\pi + z) = \frac{S_T(z) S_R(\mathrm{i}\pi - z)}{S_R(z) S_T(\mathrm{i}\pi - z)}\,. \tag{8.41}$$

The ratio S_T/S_R can be obtained by solving (8.24), that gives

$$\frac{S_T(\theta)}{S_R(\theta)} = \frac{\sinh \lambda\theta}{\sinh \lambda \mathrm{i}\pi}\,. \tag{8.42}$$

Again, λ is a free parameter that can be fixed to $\lambda = \xi/\pi$ by comparison with the known semiclassical expansion of the bound states masses [11], that will be discussed in the next Section. Then, plugging (8.42) into (8.41) and using (8.40), one easily gets (8.39).

8.1.2 Pole structure and bound-states

It can be easily seen in (8.31) that $S(\theta)$ has a set of poles in $\theta = \mathrm{i}n\xi$, for $n = 1, 2, \ldots$. On the other hand, $S_T(\theta)$ and $S_R(\theta)$ are singular respectively in $\theta = \mathrm{i}(\pi - n\xi)$ and $\theta = \mathrm{i}(\pi - n\xi), \theta = \mathrm{i}n\xi$, with $n = 1, 2, \ldots$. These poles belong to the physical strip $0 < \mathrm{Im}(\theta) < \pi$ only if $\xi < \pi$: as anticipated above, this is indeed the so-called *attractive* regime. This implies also that S has poles in the s-channel, while S_T and S_R in the t-channel. In the so-called *repulsive* regime $\xi > \pi$, instead, the poles move out of the physical strip, and therefore do not correspond to particle excitations.
If we consider the following combinations of S-matrix elements with defined charge-conjugation parity

$$S_\pm(\theta) = S_T(\theta) \pm S_R(\theta)\,, \tag{8.43}$$

then S_+ has poles in $\theta = \mathrm{i}(\pi - n\xi)$ for even n, S_- for odd n. These bound states are called *breathers*, with mass spectrum given by [53, 59]

$$m_n = 2M \sin\frac{n\xi}{2}\ ; \quad n = 1, \ldots, N\ ; \quad N < \left[\frac{\pi}{\xi}\right]\,, \tag{8.44}$$

where $[x]$ denotes the integer part of x.
The S-matrices for the bound states can be derived by defining the following ZF operators B_n

$$B_n\left(\frac{\theta_1+\theta_2}{2}\right) = [s(\theta_2)\bar{s}(\theta_1) + \bar{s}(\theta_2)s(\theta_1)]|_{\theta_1-\theta_2=\mathrm{i}(\pi-n\xi)}\ ; \quad \text{for } n \text{ even}\,, \tag{8.45}$$

$$B_n\left(\frac{\theta_1+\theta_2}{2}\right) = [s(\theta_2)\bar{s}(\theta_1) - \bar{s}(\theta_2)s(\theta_1)]|_{\theta_1-\theta_2=\mathrm{i}(\pi-n\xi)}\ ; \quad \text{for } n \text{ odd}\,, \tag{8.46}$$

that create the nth breathers. Then the bootstrap equations (5.15) can be written as commutation relations of bound state and soliton, or antisoliton, ZF generators

$$s(\theta_1)B_n(\theta_2) = S^{(n)}(\theta_{12})B_n(\theta_2)s(\theta_1)\,, \tag{8.47}$$

$$\bar{s}(\theta_1)B_n(\theta_2) = S^{(n)}(\theta_{12})B_n(\theta_2)\bar{s}(\theta_1)\,, \tag{8.48}$$

while the S-matrices for scattering between bound states are calculated by

$$B_n(\theta_1)B_m(\theta_2) = S^{(nm)}(\theta_{12})B_m(\theta_2)B_n(\theta_1)\,. \tag{8.49}$$

Alternatively, the breather-particle S-matrix can be calculated using (5.16). The projector Γ^n_{ij} is the eigenvector of the S-matrix corresponding to its singular eigenvalue [60]. Indeed, the S-matrix is diagonalized as follows

$$S^{kl}_{ij}(\theta) = \sum_e \Gamma^{kl}_e S_e(\theta)\Gamma^e_{ij}\,, \tag{8.50}$$

where S_e, with $e = 1, \ldots, 4$, are the eigenvalues and Γ^{ij}_e the corresponding eigenvectors. One of the eigenvalues turns out to be the singular combination $S_-(\theta)$ as defined in (8.43), while $\Gamma^{ij}_- = (0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$. Then one gets the following amplitude for the lowest bound state B_1:

$$\begin{aligned} S^{(1)}(\theta_1 - \theta_2) &= \frac{1}{2}\left[S_T(\theta_{13})S(\theta_{14}) - S_R(\theta_{13})S_R(\theta_{14}) + S(\theta_{13})S_T(\theta_{14})\right]\Big|_{\theta_3-\theta_4=\mathrm{i}(\pi-\xi)} \\ &= \frac{\sinh\theta_{12} + \mathrm{i}\sin\frac{\pi+\xi}{2}}{\sinh\theta_{12} - \mathrm{i}\sin\frac{\pi+\xi}{2}}\,. \end{aligned} \tag{8.51}$$

Actually, this is the only amplitude needed to describe the single breather-particle scattering, since only S_- has a pole at $\theta = \mathrm{i}(\pi - n\xi)$, for $n = 1$.
On the other hand, using (5.17), one can get the following breather-breather amplitude

$$S^{(11)}(\theta) = \frac{\sinh\theta + \mathrm{i}\sin\xi}{\sinh\theta - \mathrm{i}\sin\xi}\,, \tag{8.52}$$

whose expansion in powers of β^2 has been successfully compared to the perturbation theory for the Lagrangian (8.1), since B_1 is actually a pseudo-scalar particle corresponding to the fundamental field of sine-Gordon [61–63].

Exercises

1. Derive $S^{(1)}(\theta)$ using relation (8.47) and the identities

$$S(\theta_{32})S_R(\theta_{31}) = S_R(\theta_{32})S_T(\theta_{31})\,, \quad S(\theta_{32})S_T(\theta_{31}) - S_R(\theta_{31})S_R(\theta_{31}) = S_T(\theta_{32})S(\theta_{31})\,,$$

valid for $\theta_{12} = \mathrm{i}(\pi - \xi)$, and verify the explicit expression given in (8.51). Finally, using the fusion of two amplitudes $S^{(1)}(\theta)$, check expression (8.52).

2. Derive (8.51) using (5.16): verify that $S_-(\theta)$ in (8.43) is the only singular eigenvalue in the case of one breather, $\Gamma_-^{ij} = (0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$ and

$$\operatorname*{Res}_{\theta=\mathrm{i}(\pi-\xi)} (S_{sG})_{ij}^{kl}(\theta) \propto \Gamma_{ij}^{-}\Gamma_{-}^{kl} \,.$$

8.1.3 Form Factors

The soliton-soliton form factor of sine-Gordon satisfies the following Watson equations

$$F_{ss}(\theta) = F_{ss}(-\theta)S(\theta) = F_{ss}(2\pi\mathrm{i} - \theta) \,, \tag{8.53}$$

where $S(\theta)$ is the sine-Gordon soliton-soliton amplitude (8.39). The minimal solution of (8.53) can be found in a way analogous to the procedure, discussed in Section 8.1.1, that was used to fix the soliton-soliton amplitude of sine-Gordon as a solution of the crossing and unitarity constraints (8.26), (8.20). The result is [64]

$$F_{ss}^{min}(\theta) = -\mathrm{i}\sinh\frac{\theta}{2}\exp\left[\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{(\pi-\xi)t}{2\pi}}{\sinh\frac{\xi t}{2\pi}\cosh\frac{t}{2}}\frac{1-\cosh t\left(1-\frac{\theta}{\mathrm{i}\pi}\right)}{2\sinh t}\right] , \tag{8.54}$$

where the factor $-\mathrm{i}\sinh\frac{\theta}{2}$ is due to the overall minus sign in (8.39). The solution (8.54) can be derived in a simpler way by applying equation (7.19) to the soliton-soliton amplitude (8.39) [21]. In general, with an amplitude given by

$$S(\theta) = \exp\left[\int_0^\infty dt f(t)\sinh\frac{t\theta}{\mathrm{i}\pi}\right] , \tag{8.55}$$

the corresponding minimal solution for the form factor is

$$F^{min}(\theta) = \exp\left[\int_0^\infty dt f(t)\frac{1-\cosh t\left(1-\frac{\theta}{\mathrm{i}\pi}\right)}{2\sinh t}\right] . \tag{8.56}$$

Full expressions of soliton-soliton form factors are given then by the minimal solution (8.54) multiplied by normalization constants and factors giving additional zeros/poles in the physical strip: both of these objects depend crucially on the operator connecting the soliton-soliton state to the vacuum, as mentioned in Section 7.
For example, the breather-breather form factors are given by

$$F_{bb}^{\mathcal{O}}(\theta_{12}) = \mathcal{N}^{\mathcal{O}} K_{bb}^{\mathcal{O}}(\theta_{12}) F_{bb}^{min}(\theta_{12}) \,, \tag{8.57}$$

whose minimal solution can be derived just from the corresponding amplitude (8.52) by using (8.55), (8.56), as in the previous case. Indeed (8.52) can be written as (8.55), with

$$f(t) = 2\frac{\cosh\frac{(\pi-\xi)t}{\pi}}{t\cosh\frac{t}{2}} \,. \tag{8.58}$$

If the operator is $O(x) = \phi^2(x)$, then $K_{bb}^{\phi^2}(\theta)$ turns out to be (7.17) with $n = 1, \alpha_1 = \xi$, while $\mathcal{N}^{\phi^2}$ can be fixed by matching the large θ asymptotic behavior of (8.57) with the corresponding small-β diagrammatic perturbative result [21].

8.2 Chiral Gross-Neveu

The $SU(N)$ chiral Gross-Neveu (cGN) model [65] is described by the Lagrangian (see [31] and [8])

$$\mathcal{L}_{cGN} = \mathrm{i}\sum_{i=1}^{N}\bar{\psi}_i\partial\!\!\!/\psi_i + \frac{g_{cGN}^2}{2}\left[\left(\sum_{i=1}^{N}\bar{\psi}_i\psi_i\right)^2 - \left(\sum_{i=1}^{N}\bar{\psi}_i\gamma^5\psi_i\right)^2\right], \tag{8.59}$$

and its particle spectrum consists of $N-1$ multiplets with masses

$$m_n = m_1\frac{\sin\frac{n\pi}{N}}{\sin\frac{\pi}{N}} \quad n = 1,\dots,N-1\,. \tag{8.60}$$

The form of the S-matrix for two fundamental particles is constrained by the $SU(N)$ symmetry to be [66–70]

$$S_{ij}^{kl}(\theta) = S_0^{(N)}(\theta)\left(b_N(\theta)\delta_i^l\delta_j^k + c_N(\theta)\delta_i^k\delta_j^l\right), \tag{8.61}$$

with indices i,j,k,l running over $1,\dots,N$. The overall scalar factor and the ratio between transmission and reflection amplitudes are instead given by

$$S_0^{(N)}(\theta) = -\frac{\Gamma\left(1+\frac{\mathrm{i}\theta}{2\pi}\right)\Gamma\left(1-\frac{\mathrm{i}\theta}{2\pi}-\frac{1}{N}\right)}{\Gamma\left(1-\frac{\mathrm{i}\theta}{2\pi}\right)\Gamma\left(1+\frac{\mathrm{i}\theta}{2\pi}-\frac{1}{N}\right)}, \quad c_N(\theta) = -\frac{2\pi\mathrm{i}}{N\theta}b_N(\theta)\,, \tag{8.62}$$

that are determined by unitarity, crossing symmetry and the YBE (3.11), which in particular fixes the proportionality factor between $c_N(\theta)$ and $b_N(\theta)$.

8.2.1 Solutions for the $SU(2)$ and $SU(3)$ S-matrices

In particular, for $N=2$ the particle-particle S-matrix turns out to be the limit $\xi\to\infty$ (or $\beta^2\to 8\pi$) of the sine-Gordon S-matrix (8.18)-(8.25)-(8.31): the commutation conditions with the coproducts (6.2) built on the $SU(2)$ generators (Pauli matrices) restrict the S-matrix to be

$$S_{cGN}^{SU(2)}(\theta) = S_0^{(2)}(\theta)\begin{pmatrix} a_2(\theta) & & & \\ & b_2(\theta) & c_2(\theta) & \\ & c_2(\theta) & b_2(\theta) & \\ & & & a_2(\theta)\end{pmatrix}, \quad a_2(\theta) = b_2(\theta)+c_2(\theta)\,. \tag{8.63}$$

We consider also the $SU(3)$ case (that will be useful for [6]), whose symmetry algebra is generated by the eight Gell-Mann matrices. Imposing the commutation with four of them

is enough to fix the following structure of the S-matrix:

$$S_{cGN}^{SU(3)}(\theta) = S_0^{(3)}(\theta) \begin{pmatrix} a_3(\theta) & & & & & & & & \\ & b_3(\theta) & & c_3(\theta) & & & & & \\ & & b_3(\theta) & & & & c_3(\theta) & & \\ & c_3(\theta) & & b_3(\theta) & & & & & \\ & & & & a_3(\theta) & & & & \\ & & & & & b_3(\theta) & & c_3(\theta) & \\ & & c_3(\theta) & & & & b_3(\theta) & & \\ & & & & & c_3(\theta) & & b_3(\theta) & \\ & & & & & & & & a_3(\theta) \end{pmatrix}, \tag{8.64}$$

with $a_3(\theta) = b_3(\theta) + c_3(\theta)$. The matrix elements and the (minimal) scalar factors are determined by the YBE, unitarity and crossing symmetry, using the charge conjugation matrix

$$\mathcal{C}_{a_1 \dots a_N} = \epsilon_{a_1 \dots a_N}\,; \quad \mathcal{C}^{a_1 \dots a_N} = (-1)^{N-1} \epsilon^{a_1 \dots a_N}\,, \tag{8.65}$$

with $\epsilon_{a_1 \dots a_N}$ and $\epsilon^{a_1 \dots a_N}$ totally antisymmetric tensors. For the $SU(2)$ case the resulting elements read

$$a_2(\theta) = 1\,, \quad b_2(\theta) = \frac{\theta}{\theta - \mathrm{i}\pi}\,, \quad c_2(\theta) = \frac{-\mathrm{i}\pi}{\theta - \mathrm{i}\pi}\,, \tag{8.66}$$

while for $SU(3)$ one finds

$$a_3(\theta) = 1\,, \quad b_3(\theta) = \frac{3\theta}{3\theta - 2\mathrm{i}\pi}\,, \quad c_3(\theta) = \frac{-2\mathrm{i}\pi}{3\theta - 2\mathrm{i}\pi}\,. \tag{8.67}$$

The scalar factors are given by (8.62) with $N = 2$ and $N = 3$ respectively.

8.2.2 Pole structure and bound-states

Since the $SU(2)$ cGN S-matrix can be thought of as the $\xi \to \infty$ limit of the sine-Gordon one, and this limit corresponds to the highest repulsive regime for sine-Gordon, it can be easily understood that there are no bound states in this theory. Otherwise, it can be also verified that the S-matrix does not have any pole in the strip $0 \leq \mathrm{Im}(\theta) \leq \pi$.

On the other hand, the $SU(3)$ S-matrix has a pole in the physical strip at $\theta = \frac{2\mathrm{i}\pi}{3}$, corresponding to a bound state with mass $m_2 = m_1$, then equal to the fundamental particle mass. For generic N, bound states have masses given by the expression (8.60), so that $m_{N-1} = m_1$. Indeed, it is possible to show that in the $SU(N)$ cGN the antiparticles are $N-1$ bound states of particles and vice versa [67].

We can now derive the particle-bound state S-matrix using (5.15) and finding Γ_{ij}^n, given by the antisymmetric tensor ϵ_{nij}, as the three eigenvectors corresponding to the singular

eigenvalue $b(\theta) - c(\theta)$. The result is

$$S^{bp}_{cGN}(\theta) = S^{bp}_0(\theta) \begin{pmatrix} A(\theta) & & & & C(\theta) & & & C(\theta) \\ & B(\theta) & & & & & & \\ & & B(\theta) & & & & & \\ & & & B(\theta) & & & & \\ C(\theta) & & & & A(\theta) & & & C(\theta) \\ & & & & & B(\theta) & & \\ & & & & & & B(\theta) & \\ & & & & & & & B(\theta) \\ C(\theta) & & & & C(\theta) & & & A(\theta) \end{pmatrix}, \tag{8.68}$$

where $A(\theta) = B(\theta) + C(\theta)$, the scalar factor is given by the fusion of the fundamental ones

$$S^{bp}_0(\theta) = S^{(3)}_0(\theta - \mathrm{i}\pi/3) S^{(3)}_0(\theta + \mathrm{i}\pi/3) = \frac{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{7}{6} + \mathrm{i}\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{\theta}{2\pi}\right)\Gamma\left(\frac{7}{6} - \mathrm{i}\frac{\theta}{2\pi}\right)}, \tag{8.69}$$

and the remaining matrix elements read

$$\begin{aligned} B(\theta) = \frac{1}{2} \operatorname*{Res}_{\theta_{12}=2\mathrm{i}\pi/3} & \left(b(\theta_{12}) - c(\theta_{12})\right)\left(2b(\theta_{13})b(\theta_{23}) + b(\theta_{13})c(\theta_{23}) + c(\theta_{13})b(\theta_{23})\right. \\ & \left. - c(\theta_{13})c(\theta_{23})\right), \end{aligned} \tag{8.70}$$

$$C(\theta) = -\frac{1}{2} \operatorname*{Res}_{\theta_{12}=2\mathrm{i}\pi/3} \left(b(\theta_{12}) - c(\theta_{12})\right)\left(b(\theta_{13})c(\theta_{23}) + c(\theta_{13})b(\theta_{23}) - c(\theta_{13})c(\theta_{23})\right). \tag{8.71}$$

Finally, the bound state-bound state amplitudes can be derived using (5.17): we leave this as an exercise to the interested reader.

8.2.3 Form Factors

Here we show the simplest example of form factors in $SU(N)$ cGN models, that is the minimal solution to the 2-particle Watson equations (8.53), corresponding to the amplitudes $a_{2,3}(\theta) = 1$ of the S-matrices (8.63) and (8.64). Then the amplitudes are actually $S^{(N)}_0$ as written in (8.62) and, using the trick given by equations (8.55) and (8.56), one can easily find

$$F^{min}_{11}(\theta) = c \exp\left[\int_0^\infty \frac{dt}{t} \frac{e^{\frac{t}{N}} \sinh t\left(1 - \frac{1}{N}\right)}{\sinh^2 t}\left(1 - \cosh t\left(1 - \frac{\theta}{\mathrm{i}\pi}\right)\right)\right]. \tag{8.72}$$

A generic n-particle form factor for scalar operators would be

$$F^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) = K^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) \prod_{1 \le i < j \le n} F^{min}_{a_i a_j}(\theta_{ij}), \tag{8.73}$$

where the function $K^{\mathcal{O}}_{\underline{a}}$ contains the pole structure and is partially fixed by the Watson equations (7.10)-(7.11), with the amplitude S replaced by $\tilde{S} = S/S_0$ [71]:

$$\begin{aligned} & K^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_i, \theta_j, \dots, \theta_n) = K^{\mathcal{O}}_{a_1, \dots, a_j, a_i, \dots, a_n}(\theta_1, \dots, \theta_j, \theta_i, \dots, \theta_n) \tilde{S}_{a_i a_j}(\theta_{ij}), \quad j = i + 1, \\ & K^{\mathcal{O}}_{\underline{a}}(\theta_1 + 2\pi i, \dots, \theta_n) = K^{\mathcal{O}}_{\underline{a}}(\theta_1, \dots, \theta_n) \prod_{i=2}^{n} \tilde{S}_{a_i a_1}(\theta_i - \theta_1). \end{aligned} \tag{8.74}$$

The only solutions of these equations we found in literature for $SU(N)$ cGN were obtained using the so-called "off-shell" nested Bethe ansatz method (see [24, 22, 71, 72] for example), that is beyond the scope of these lectures.

8.3 AdS/CFTs

In AdS_5/CFT_4, the dynamics of string excitations is described by an integrable non-linear σ-model defined on the super coset $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ [74], while, on the gauge side, the fields composing single-trace operators correspond to the excitations of an integrable super spin chain [75] (see also the aforementioned reviews [12] and [20]).
Then such excitations interact via a factorized S-matrix, depending on the 't Hooft coupling λ, whose matrix elements were fixed in [16], up to an overall scalar factor, by imposing the invariance of the S-matrix under two copies of the centrally extended $su(2|2)$ superalgebra, that is the symmetry algebra leaving invariant the vacuum.
In order to determine the scalar factor, crossing symmetry has been imposed in the algebraic ways explained in Sections 4.2.1 and 6, in [19] and [17] respectively. The equation arising from such condition was satisfied by the conjecture of [18] and was finally solved in [37] (see also [14] for a review).
The bound states S-matrices have been determined in [76] by using the Yangian symmetry $Y(su(2|2))$ [31]. Recently, the usual bootstrap procedure was generalized to the AdS_5/CFT_4 case [77].
Determining the exact, all-loop S-matrix in AdS_5/CFT_4 has been of essential importance, as in any other integrable theory, to study its exact finite volume spectrum. From the S-matrix of [16], indeed, the asymptotic Bethe equations conjectured in [79] could be derived [16, 78]. Then, on the basis of the same S-matrix, it was possible to study and compute the leading order finite-size corrections [80] and the exact spectrum via the TBA [81], that was recently reduced to a simple set of non-linear Riemann-Hilbert equations in [82], the so-called Quantum Spectral Curve (QSC) equations.
Concerning AdS_4/CFT_3, the exact S-matrix was determined on the basis of a symmetry superalgebra still related to $su(2|2)$, while the scalar factors were fixed by slightly different crossing symmetry relations [83]. This S-matrix gives the Bethe equations conjectured in [84] and was used to derive Lüscher-like corrections [85] and the corresponding TBA [86] (see also the review [87]). Finally, also in this example of integrable AdS/CFT correspondence, it was possible to reduce the spectral problem to the solution of QSC equations [88].
In the case of AdS_3/CFT_2, two string backgrounds were studied, $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$, and both of them involve massless string modes, a new feature compared to AdS_5 and AdS_4. A set of all-loop Bethe equations for the massive modes of $AdS_3 \times S^3 \times S^3 \times S^1$ were conjectured in [90] and later derived from the S-matrix proposed in [89]. However, this S-matrix could describe only a sector of the theory. Imposing the commutation with the generators of the full (centrally extended $su(1|1)^2$) symmetry algebra allowed [91] to determine the complete S-matrix for massive excitations and the consequent all-loop Bethe equations [92] describing the large volume limit of the $AdS_3 \times S^3 \times S^3 \times S^1$ massive spectrum. Massless modes were included in the integrability framework in [93],

while for the $AdS_3 \times S^3 \times T^4$ case the reader can look at the complete S-matrix determined in [94]. These S-matrices are substantially more involved than the ones appearing in the higher-dimensional holographic pairs, due to the presence of several distinct scalar factors with novel properties. An all-loop proposal exists for the scalar factors, involving both massive [95] and massless [96] modes, of the $AdS_3 \times S^3 \times T^4$ S-matrix. Furthermore, finite-size corrections due to massless modes seem to play a new rôle in the calculation of the large volume spectrum [97]. See [98] for a review about these and other developments of integrability in AdS_3/CFT_2.
Finally, in AdS_2/CFT_1 the determination of an exact S-matrix and related Bethe equations is even more difficult due to the presence of more massless modes and less supersymmetry, while crossing symmetry relations are still understood only formally and it is not clear what is the CFT_1 involved in this duality [99].

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References

[1] J. A. Wheeler, "On the mathematical description of light nuclei by the method of resonating group structure", Phys. Rev. **52** (1937) 1107.

[2] W. Heisenberg, "Der mathematische Rahmen der Quantentheorie der Wellenfelder", Zeit. für Naturforschung **1** (1946) 608.

[3] G. Chew, "The S-matrix theory of strong interaction", W. A. Benjamin Inc., New York (1961).

[4] S. Mandelstam, "Determination of the pion - nucleon scattering amplitude from dispersion relations and unitarity. General theory", Phys. Rev. **112** (1958) 1344.

[5] S. R. Coleman and J. Mandula, "All possible symmetries of the S-matrix", Phys. Rev. **159** (1967) 1251.

[6] F. Levkovich-Maslyuk, "Lectures on the Bethe Ansatz", in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[7] M. Lüscher, "On a relation between finite size effects and elastic scattering processes", in the proceedings of the Cargese Summer Institute: "Progress in gauge field theory", edited by G. 't Hooft, A. Singer, R. Stora, Plenum Press (1984); M. Lüscher, "Volume dependence of the energy spectrum in massive quantum field theories. 1. Stable particle states", Commun. Math. Phys. **104** (1986) 177; T. R. Klassen and E. Melzer, "On the relation between scattering amplitudes and finite size mass corrections in QFT", Nucl. Phys. B **362** (1991) 329.

[8] S. J. van Tongeren, "Introduction to the Thermodynamic Bethe Ansatz", in "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[9] G. Mussardo, "Statistical field theory, an introduction to exactly solved models in statistical physics", Oxford University Press (1992).

[10] P. Dorey, "Exact S-matrices", in the proceedings of the Eötvös Summer School in Physics: "Conformal field theories and integrable models", edited by Z. Horvath, L. Palla, Springer (1997) [hep-th/9810026].

[11] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized S-matrices in two-dimensions as the exact solutions of certain relativistic quantum field models", Annals Phys. **120** (1979) 253.

[12] G. Arutyunov and S. Frolov, "Foundations of the $AdS_5 \times S^5$ superstring. Part I", J. Phys. A **42** (2009) 254003 [arXiv:0901.4937 [hep-th]].

[13] C. Ahn and R. I. Nepomechie, "Review of AdS/CFT integrability, Chapter III.2: Exact worldsheet S-matrix", Lett. Math. Phys. **99** (2012) 209 [arXiv:1012.3991 [hep-th]].

[14] P. Vieira and D. Volin, "Review of AdS/CFT Integrability, Chapter III.3: The Dressing factor," Lett. Math. Phys. **99** (2012) 231 [arXiv:1012.3992 [hep-th]].

[15] M. Staudacher, "The factorized S-matrix of CFT/AdS", JHEP **0505** (2005) 054 [hep-th/0412188].

[16] N. Beisert, "The SU(2|2) dynamic S-matrix", Adv. Theor. Math. Phys. **12** (2008) 945 [hep-th/0511082]; N. Beisert, "The analytic Bethe Ansatz for a chain with centrally extended su(2|2) Symmetry", J. Stat. Mech. **0701** (2007) P01017 [nlin/0610017 [nlin.SI]].

[17] R. A. Janik, "The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry", Phys. Rev. D **73** (2006) 086006 [hep-th/0603038].

[18] G. Arutyunov, S. Frolov and M. Staudacher, "Bethe Ansatz for quantum strings", JHEP **0410** (2004) 016 [hep-th/0406256]; N. Beisert, B. Eden and M. Staudacher, "Transcendentality and crossing", J. Stat. Mech. **0701** (2007) P01021 [hep-th/0610251].

[19] G. Arutyunov, S. Frolov and M. Zamaklar, "The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring", JHEP **0704** (2007) 002 [hep-th/0612229].

[20] N. Beisert *et al.*, "Review of AdS/CFT integrability: an overview", Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982 [hep-th]].

[21] M. Karowski and P. Weisz, "Exact form factors in (1+1)-dimensional field theoretic models with soliton behavior", Nucl. Phys. B **139** (1978) 455.

[22] H. M. Babujian, A. Foerster and M. Karowski, "The form factor program: A review and new results: The nested SU(N) off-shell Bethe Ansatz", SIGMA **2** (2006) 082 [hep-th/0609130].

[23] V. P. Yurov and A. B. Zamolodchikov, "Correlation functions of integrable 2-D models of relativistic field theory. Ising model", Int. J. Mod. Phys. A **6** (1991) 3419.

[24] H. M. Babujian, A. Fring, M. Karowski and A. Zapletal, "Exact form-factors in integrable quantum field theories: The sine-Gordon model", Nucl. Phys. B **538** (1999) 535 [hep-th/9805185].

[25] F. A. Smirnov,"Form factors in completely integrable models of quantum field theory", Adv. Series in Math. Phys. **14**, World Scientific (1992).

[26] A. M. Polyakov, "Hidden symmetry of the two-dimensional chiral fields", Phys. Lett. B **72** (1977) 224.

[27] R. Shankar and E. Witten, "The S-matrix of the supersymmetric nonlinear sigma model", Phys. Rev. D **17** (1978) 2134.

[28] S. J. Parke, "Absence of particle production and factorization of the S-matrix in (1+1)-dimensional models", Nucl. Phys. B **174** (1980) 166.

[29] D. Iagolnitzer, "Factorization of the multiparticle S-matrix in two-dimensional space-time models", Phys. Rev. D **18** (1978) 1275.

[30] M. Lüscher, "Quantum nonlocal charges and absence of particle production in the two-dimensional nonlinear sigma model", Nucl. Phys. B **135** (1978) 1.

[31] F. Loebbert, "Lectures on Yangian Symmetry", in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[32] C. N. Yang, "Some exact results for the many-body problems in one dimension with repulsive delta-function interaction", Phys. Rev. Lett. **19** (1967) 1312.

[33] R. J. Baxter, "Partition function of the eight vertex lattice model", Annals Phys. **70** (1972) 193, Annals Phys. **281** (2000) 187.

[34] J. L. Miramontes, "Hermitian analyticity versus real analyticity in two-dimensional factorized S-matrix theories", Phys. Lett. B **455** (1999) 231 [hep-th/9901145].

[35] L. D. Faddeev, "Quantum completely integrable models of field theory", Sov. Sci. Rev. C **1** (1980) 107.

[36] L. Castillejo, R. H. Dalitz and F. J. Dyson, "Low's scattering equation for the charged and neutral scalar theories", Phys. Rev. **101** (1956) 453.

[37] D. Volin, "Minimal solution of the AdS/CFT crossing equation", J. Phys. A **42** (2009) 372001 [arXiv:0904.4929 [hep-th]].

[38] T. R. Klassen and E. Melzer, "Purely elastic scattering theories and their ultraviolet limits", Nucl. Phys. B **338** (1990) 485.

[39] F. Spill, "Hopf algebras in the AdS/CFT correspondence", Diploma thesis at Humboldt University, Berlin (2007).

[40] V. G. Drinfeld, "Quantum groups", J. Sov. Math. **41** (1988) 898, Zap. Nauchn. Semin. **155** (1986) 18.

[41] G. W. Delius, "Exact S-matrices with affine quantum group symmetry", Nucl. Phys. B **451** (1995) 445 [hep-th/9503079].

[42] N. J. MacKay, "On the algebraic structure of factorized S-matrices", Durham theses, Durham University (1992), http://etheses.dur.ac.uk/5764/.

[43] F. Spill, "Yangians in integrable field theories, spin chains and gauge-string dualities", Rev. Math. Phys. **24** (2012) 1230001.

[44] O. Babelon and D. Bernard, "From form-factors to correlation functions: The Ising model", Phys. Lett. B **288** (1992) 113 [hep-th/9206003].

[45] B. Pozsgay and G. Takacs, "Form factors in finite volume. II. Disconnected terms and finite temperature correlators", Nucl. Phys. B **788** (2008) 209 [arXiv:0706.3605 [hep-th]].

[46] H. Lehmann, K. Symanzik and W. Zimmermann, "On the formulation of quantized field theories", Nuovo Cim. **1** (1955) 205.

[47] K. M. Watson, "Some general relations between the photoproduction and scattering of pi mesons", Phys. Rev. **95** (1954) 228.

[48] L. D. Faddeev and L. A. Takhtajan, "Essentially nonlinear one-dimensional model of the classical field theory", Theor. Math. Phys. **21** (1975) 1046, Teor. Mat. Fiz. **21** (1974) 160.

[49] P. P. Kulish and E. R. Nissimov, "Conservation laws in the quantum theory: $\cos\phi$ in two-dimensions and in the massive Thirring model", JETP Lett. **24** (1976) 220, Pisma Zh. Eksp. Teor. Fiz. **24** (1976) 247.

[50] S. R. Coleman, "The quantum sine-Gordon equation as the massive Thirring model", Phys. Rev. D **11** (1975) 2088.

[51] S. Mandelstam, "Soliton operators for the quantized sine-Gordon equation", Phys. Rev. D **11** (1975) 3026.

[52] T. R. Klassen and E. Melzer, "Sine-Gordon not equal to massive Thirring, and related heresies", Int. J. Mod. Phys. A **8** (1993) 413 [hep-th/9206114].

[53] V. E. Korepin and L. D. Faddeev, "Quantization of solitons", Theor. Math. Phys. **25** (1975) 1039, Teor. Mat. Fiz. **25** (1975) 147.

[54] V. E. Korepin, P. P. Kulish and L. D. Faddeev, "Soliton quantization", JETP Lett. **21** (1975) 138, Pisma Zh. Eksp. Teor. Fiz. **21** (1975) 302.

[55] A. B. Zamolodchikov, "Exact S-matrix of quantum sine-Gordon solitons", JETP Lett. **25** (1977) 468.

[56] R. Jackiw and G. Woo, "Semiclassical scattering of quantized nonlinear waves", Phys. Rev. D **12** (1975) 1643.

[57] M. Karowski, H. J. Thun, T. T. Truong and P. H. Weisz, "On the uniqueness of a purely elastic S-matrix in (1+1)-dimensions", Phys. Lett. B **67** (1977) 321.

[58] M. Karowski, "An exact relativistic S-matrix in (1+1)-dimensions: The on-shell solution of the massive Thirring model and the quantum sine-Gordon equation", FUB-HEP-77-19A.

[59] R. F. Dashen, B. Hasslacher and A. Neveu, "The particle spectrum in model field theories from semiclassical functional integral techniques", Phys. Rev. D **11** (1975) 3424.

[60] M. Karowski and H. J. Thun, "Complete S-matrix of the massive Thirring model", Nucl. Phys. B **130** (1977) 295.

[61] I. Arefeva and V. Korepin, "Scattering in two-dimensional model with Lagrangian $1/\gamma((d_\mu u)^2/2 + m^2\cos(u-1))$", Pisma Zh. Eksp. Teor. Fiz. **20** (1974) 680.

[62] S. N. Vergeles and V. M. Gryanik, "Two-dimensional quantum field theories having exact solutions", Sov. J. Nucl. Phys. **23** (1976) 704, Yad. Fiz. **23** (1976) 1324.

[63] B. Schroer, T. T. Truong and P. Weisz, "Towards an explicit construction of the sine-Gordon field theory", Phys. Lett. B **63** (1976) 422.

[64] P. H. Weisz, "Exact quantum sine-Gordon soliton form factors", Phys. Lett. B **67** (1977) 179.

[65] D. J. Gross and A. Neveu, "Dynamical symmetry breaking in asymptotically free field theories", Phys. Rev. D **10** (1974) 3235.

[66] B. Berg, M. Karowski, P. Weisz and V. Kurak, "Factorized U(n) symmetric S-matrices in two-dimensions", Nucl. Phys. B **134** (1978) 125.

[67] V. Kurak and J. A. Swieca, "Anti-particles as bound states of particles in the factorized S-matrix framework", Phys. Lett. B **82** (1979) 289.

[68] B. Berg and P. Weisz, "Exact S-matrix of the chiral invariant SU(N) Thirring model", Nucl. Phys. B **146** (1978) 205.

[69] R. Koberle, V. Kurak and J. A. Swieca, "Scattering theory and 1/N expansion in the chiral Gross-Neveu model", Phys. Rev. D **20** (1979) 897.

[70] E. Abdalla, B. Berg and P. Weisz, "More about the S-matrix of the chiral SU(N) Thirring model", Nucl. Phys. B **157** (1979) 387.

[71] H. M. Babujian, A. Foerster and M. Karowski, "The nested SU(N) off-shell Bethe Ansatz and exact form-factors", J. Phys. A **41** (2008) 275202

[72] H. Babujian, A. Foerster and M. Karowski, "Exact form factors of the SU(N) Gross-Neveu model and 1/N expansion", Nucl. Phys. B **825** (2010) 396 [arXiv:0907.0662 [hep-th]].

[73] A. Torrielli, "Lectures on Classical Integrability", in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. M. Szécsényi and S. J. van Tongeren, Journal of Physics A (2016), to appear.

[74] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the $AdS_5 \times S^5$ superstring", Phys. Rev. D **69** (2004) 046002 [hep-th/0305116].

[75] J. A. Minahan and K. Zarembo, "The Bethe Ansatz for $\mathcal{N}$=4 Super-Yang-Mills", JHEP **0303** (2003) 013 [hep-th/0212208].

[76] G. Arutyunov, M. de Leeuw and A. Torrielli, "The bound state S-matrix for $AdS_5 \times S^5$ superstring", Nucl. Phys. B **819** (2009) 319 [arXiv:0902.0183 [hep-th]].

[77] N. Beisert, M. de Leeuw and P. Nag, "Fusion for the one-dimensional Hubbard model", J. Phys. A **48** (2015) 32, 324002 [arXiv:1503.04838 [math-ph]].

[78] M. J. Martins and C. S. Melo, "The Bethe Ansatz approach for factorizable centrally extended su(2|2) S-matrices", Nucl. Phys. B **785** (2007) 246 [hep-th/0703086].

[79] N. Beisert and M. Staudacher, "Long-range psu(2,2|4) Bethe ansätze for gauge theory and strings", Nucl. Phys. B **727** (2005) 1 [hep-th/0504190].

[80] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling: The giant magnon", Phys. Rev. D **76** (2007) 126008 [arXiv:0708.2208 [hep-th]]; Z. Bajnok and R. A. Janik, "Four-loop perturbative Konishi from strings and finite size effects for multiparticle states", Nucl. Phys. B **807** (2009) 625 [arXiv:0807.0399 [hep-th]].

[81] D. Bombardelli, D. Fioravanti and R. Tateo, "Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal", J. Phys. A **42**, 375401 (2009) [arXiv:0902.3930 [hep-th]]; N. Gromov, V. Kazakov, A. Kozak and P. Vieira, "Exact spectrum of anomalous dimensions of planar $\mathcal{N}$= 4 supersymmetric Yang-Mills theory: TBA and excited states", Lett. Math. Phys. **91** (2010) 265 [arXiv:0902.4458 [hep-th]]; G. Arutyunov and S. Frolov, "Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ mirror model", JHEP **0905** (2009) 068 [arXiv:0903.0141 [hep-th]].

[82] N. Gromov, V. Kazakov, S. Leurent and D. Volin, "Quantum Spectral Curve for planar $\mathcal{N}$=4 super-Yang-Mills theory", Phys. Rev. Lett. **112** (2014) 1, 011602 [arXiv:1305.1939 [hep-th]]; N. Gromov, V. Kazakov, S. Leurent and D. Volin, "Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4", JHEP **1509** (2015) 187 [arXiv:1405.4857 [hep-th]].

[83] C. Ahn and R. I. Nepomechie, "$\mathcal{N}$=6 super Chern-Simons theory S-matrix and all-loop Bethe ansatz equations", JHEP **0809** (2008) 010 [arXiv:0807.1924 [hep-th]].

[84] N. Gromov and P. Vieira, "The all loop AdS_4/CFT_3 Bethe Ansatz", JHEP **0901** (2009) 016 [arXiv:0807.0777 [hep-th]].

[85] D. Bombardelli and D. Fioravanti, "Finite-size corrections of the CP^3 giant magnons: The Lüscher terms", JHEP **0907** (2009) 034 [arXiv:0810.0704 [hep-th]]; T. Lukowski and O. Ohlsson Sax, "Finite size giant magnons in the SU(2)×SU(2) sector of $AdS_4 \times CP^3$", JHEP **0812** (2008) 073 [arXiv:0810.1246 [hep-th]]; C. Ahn and P. Bozhilov, "Finite-size effect of the dyonic giant magnons in $\mathcal{N}$=6 super Chern-Simons theory", Phys. Rev. D **79** (2009) 046008 [arXiv:0810.2079 [hep-th]]; M. C. Abbott, I. Aniceto and D. Bombardelli, "Real and virtual bound states in Lüscher corrections for CP^3 magnons", J. Phys. A **45** (2012) 335401 [arXiv:1111.2839 [hep-th]].

[86] D. Bombardelli, D. Fioravanti and R. Tateo, "TBA and Y-system for planar AdS_4/CFT_3", Nucl. Phys. B **834** (2010) 543 [arXiv:0912.4715 [hep-th]]; N. Gromov and F. Levkovich-Maslyuk, "Y-system, TBA and Quasi-Classical strings in $AdS_4 \times CP^3$", JHEP **1006** (2010) 088 [arXiv:0912.4911 [hep-th]].

[87] T. Klose, "Review of AdS/CFT integrability, Chapter IV.3: $\mathcal{N}$=6 Chern-Simons and strings on $AdS_4 \times CP^3$", Lett. Math. Phys. **99** (2012) 401 [arXiv:1012.3999 [hep-th]].

[88] A. Cavaglià, D. Fioravanti, N. Gromov and R. Tateo, "Quantum Spectral Curve of the $\mathcal{N}$=6 Supersymmetric Chern-Simons Theory", Phys. Rev. Lett. **113** (2014) 2, 021601 [arXiv:1403.1859 [hep-th]].

[89] C. Ahn and D. Bombardelli, "Exact S-matrices for AdS_3/CFT_2", Int. J. Mod. Phys. A **28** (2013) 1350168 [arXiv:1211.4512 [hep-th]].

[90] A. Babichenko, B. Stefanski, Jr. and K. Zarembo, "Integrability and the AdS_3/CFT_2 correspondence", JHEP **1003** (2010) 058 [arXiv:0912.1723 [hep-th]].

[91] R. Borsato, O. Ohlsson Sax and A. Sfondrini, "A dynamic $su(1|1)^2$ S-matrix for AdS_3/CFT_2", JHEP **1304** (2013) 113 [arXiv:1211.5119 [hep-th]].

[92] R. Borsato, O. Ohlsson Sax and A. Sfondrini, "All-loop Bethe Ansatz equations for AdS_3/CFT_2", JHEP **1304** (2013) 116 [arXiv:1212.0505 [hep-th]].

[93] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, "The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S-matrix", J. Phys. A **48** (2015) 41, 415401 [arXiv:1506.00218 [hep-th]].

[94] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, "Towards the all-loop worldsheet S-matrix for $AdS_3 \times S^3 \times T^4$", Phys. Rev. Lett. **113** (2014) 13, 131601 [arXiv:1403.4543 [hep-th]]; R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefanski, "The complete $AdS_3 \times S^3 \times T^4$ worldsheet S-matrix", JHEP **1410** (2014) 66 [arXiv:1406.0453 [hep-th]];

[95] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefanski, Jr. and A. Torrielli, "Dressing phases of AdS_3/CFT_2", Phys. Rev. D **88** (2013) 066004 [arXiv:1306.2512 [hep-th]].

[96] R. Borsato, O. O. Sax, A. Sfondrini and B. Stefanski, "On the spectrum of $AdS_3\times S^3\times T^4$ strings with Ramond-Ramond flux", arXiv:1605.00518 [hep-th].

[97] M. C. Abbott and I. Aniceto, "Massless Lüscher terms and the limitations of the AdS_3 asymptotic Bethe Ansatz", arXiv:1512.08761 [hep-th].

[98] A. Sfondrini, "Towards integrability for AdS_3/CFT_2", J. Phys. A **48** (2015) 2, 023001 [arXiv:1406.2971 [hep-th]].

[99] B. Hoare, A. Pittelli and A. Torrielli, "Integrable S-matrices, massive and massless modes and the $AdS_2\times S^2$ superstring", JHEP **1411** (2014) 051 [arXiv:1407.0303 [hep-th]]; B. Hoare, A. Pittelli and A. Torrielli, "The S-matrix algebra of the $AdS_2\times S^2$ superstring", arXiv:1509.07587 [hep-th].

Lectures on the Bethe Ansatz

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Abstract

We give a pedagogical introduction to the Bethe ansatz techniques in integrable QFTs and spin chains. We first discuss and motivate the general framework of asymptotic Bethe ansatz for the spectrum of integrable QFTs in large volume, based on the exact S-matrix. Then we illustrate this method in several concrete theories. The first case we study is the $SU(2)$ chiral Gross-Neveu model. We derive the Bethe equations via algebraic Bethe ansatz, solving in the process the Heisenberg XXX spin chain. We discuss this famous spin chain model in some detail, covering in particular the coordinate Bethe ansatz, some properties of Bethe states, and the classical scaling limit leading to finite-gap equations. Then we proceed to the more involved $SU(3)$ chiral Gross-Neveu model and derive the Bethe equations using nested algebraic Bethe ansatz to solve the arising $SU(3)$ spin chain. Finally we show how a method similar to the Bethe ansatz works in a completley different setting, namely for the 1d oscillator in quantum mechanics.

This article is part of a collection of introductory reviews originating from lectures given at the YRIS summer school in Durham during July 2015.

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Contents

1 Introduction

Integrable QFTs in 1+1 dimensions have a vast array of remarkable properties. In particular, it is often possible to exactly compute the S-matrix of the theory (an extensive discussion of this aspect is given in another part of the present collection [1]). The S-matrix captures a lot of the theory's dynamics but only describes the scattering of *asymptotic* states, i.e. particles starting from infinitely far away and then flying off to infinity again. In contrast, another interesting setup to consider is when our theory is put into a spatial box of *finite* size L. In finite volume the spectrum of the Hamiltonian becomes discrete, so a natural question to ask is what are the energies of the states.

It turns out that for integrable theories this energy spectrum can be computed to a large extent using only the scattering data. For large volume L one can write down equations for the spectrum in terms of the exact S-matrix. These equations are known as the asymptotic Bethe ansatz equations and they will be the main topic of this article. Like the S-matrix itself, they are exact at any value of the coupling constants. These equations are only valid when L is large (in a sense that will be made more precise later), but still provide a lot of important information. They are also the first step towards formulating the so-called Thermodynamic Bethe ansatz (TBA) equations which give the energies exactly at *any* L including all corrections. The TBA approach is covered in detail in a different part of this collection [2].

Importantly, both the asymptotic Bethe ansatz and the TBA have been crucial for the recent applications of integrability to several AdS/CFT dualities between gauge and string theories [3]. A key problem in this setting is computing the energies of multiparticle string states in finite volume, which are mapped to operator conformal dimensions in gauge theory (in fact the volume L corresponds to the number of elementary fields in the operator). Integrability methods have led to great success in exploring this problem, and in particular in the computation of superstring energies on the $AdS_5 \times S^5$ space which coincide with operator dimensions in the dual $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. We hope that several simpler examples discussed in this article will serve as a starting ground for understanding how Bethe Ansatz works in the AdS/CFT context.

The name 'Bethe ansatz equations' originates from the famous solution of the XXX spin chain by Hans Bethe in [4]. While our main goal is to study integrable QFTs we will see that often computing the spectrum of some QFT model leads to an auxiliary spin chain which should be solved first. We will see several examples and discuss the Bethe ansatz solutions of these spin chains as well. At the same time, various spin chain models are interesting on their own as many of them find important applications in condensed matter physics.

We will first discuss the asymptotic Bethe equations in a general setting and then cover several examples for particular models. The presentation is structured as follows. In section 2 we give physical motivation for the asymptotic Bethe ansatz in integrable QFT and write the Bethe equations in a generic form as a periodicity condition on the wavefunction. In section 3 we present in a general form the algebraic Bethe ansatz approach allowing to greatly simplify the periodicity constraint by reducing it to a transparent diagonalization problem. In section 4 we demonstrate the method in action on the example of the $SU(2)$

chiral Gross-Neveu model. In the process we obtain the solution of the celebrated XXX $SU(2)$ Heisenberg spin chain. In section 5 we discuss some features of the XXX chain and its Bethe eigenstates, as well as the classical limit leading to finite-gap equations. In section 6 we proceed to the more complicated case of the $SU(3)$ chiral Gross-Neveu model which we solve via *nested* algebraic Bethe ansatz. Finally in section 7 we illustrate the versatility of Bethe ansatz by applying a Bethe-like method to solve the 1d quantum mechanical oscillator. Some exercises for the interested reader are also included throughout the text.

There is certainly a large literature on the subject available, in particular we would like to point out several reviews discussing various aspects of the Bethe ansatz methods [5, 6, 7, 8, 9, 10]. For reasons of presentation clarity, only some selected references are included in this pedagogical article[1].

2 Asymptotic Bethe ansatz equations in 2d integrable QFTs

In this section we will discuss the Bethe equations for the spectrum of a generic 1+1 dimensional integrable theory. We will always consider a theory on a circle[2] of length L, i.e. we impose periodic boundary conditions.

As discussed in the part of this collection of articles dedicated to integrable S-matrices [1], in an integrable theory the scattering has several remarkable features:

- The number of particles is conserved in any scattering event
- The momenta do not change but can only be redistributed between particles
- The S-matrix for multiparticle scattering factorizes, i.e. the S-matrix for any number of particles is a product of the $2 \to 2$ S-matrices

Although in general one cannot introduce a wavefunction in QFT due to the production of virtual particles, for integrable theories these special features (most importantly the first one) make it possible to do this at least in some regions of the configuration space. Then from the periodicity of the wavefunction one can derive quantization conditions which determine the spectrum.

Let us first discuss a toy model – a theory with only one particle in its spectrum. An example is the sinh-Gordon model for some values of the parameters. Then an intuitive picture which gives the correct equations for the spectrum is the following one (for a more rigorous discussion see e.g. [11]). Since the number of particles is conserved we can speak of a wavefunction as in quantum mechanics. If we have n particles on a circle the wavefunction must be periodic. Imagine that we take the first particle around the circle once, eventually bringing it back to its place again.

[1] We also tried to make the notation in this article maximally consistent with other parts of the present

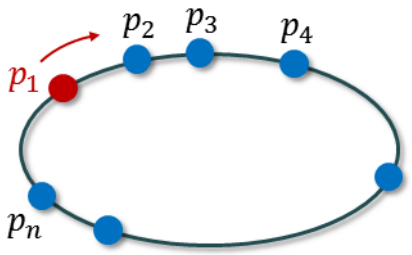

Figure 1: Deriving the periodicity condition on a circle. We take the first particle with momentum p_1 around the circle, scattering it through all the other particles.

If there were no other particles (or if the theory was non-interacting) the wavefunction would acquire a phase factor $e^{ip_1 L}$ where p_1 is the particle's momentum. Then from periodicity of the wavefunction

$$e^{ip_1 L} = 1 \ , \tag{2.1}$$

i.e. the momentum would be quantized according to

$$p_1 = \frac{2\pi k}{L}, \quad k \in \mathbb{Z} \ . \tag{2.2}$$

However we need to take into account the interaction with other particles. If L is large compared to the interaction range between particles (e.g. the inverse mass scale of the theory), the particles are almost always well separated from each other. Because of this their interaction is described by the asymptotic S-matrix which we know. The number of particles does not change in this interaction. Thus when we take a particle around the circle, it will scatter through all the other particles, and all that happens is that for each scattering the wavefunction is multiplied by the S-matrix which is just some phase factor, $S(p_1, p_2) = e^{i\alpha(p_1,p_2)}$. This product of S-matrix phases will combine with the $e^{ip_1 L}$ phase acquired due to free propagation. Thus periodicity of the wavefunction will be ensured if

$$e^{ip_1 L} S(p_1, p_2) S(p_1, p_3) \dots S(p_1, p_n) = 1 \tag{2.3}$$

Similarly for any particle we get

$$e^{ip_j L} \prod_{k=1,\, k\neq j}^{n} S(p_j, p_k) = 1 \tag{2.4}$$

This is a set of n algebraic equations for n variables $p_1 \dots, p_n$ which thus allow us to fix the values of the momenta. The system (2.4) are the asymptotic Bethe ansatz equations for our toy model.

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review volume.

[2]By a circle we actually mean a straight line segment $[0, L]$ whose endpoints $x = 0$ and $x = L$ are identified.

Once the momenta are found from (2.4), one can compute the energy of the state. In regions where the particles are well separated they propagate freely, so the energy of the state should always be equal to simply the sum of individual particles' energies. If the energy of a single particle with momentum p is $\epsilon(p)$ we will have

$$E = \sum_i \epsilon(p_i). \tag{2.5}$$

From the derivation it is clear that the asymptotic Bethe equations will describe the energy only in a large volume L. It turns out that, more precisely, the Bethe equations capture all terms in the large L expansion of the energy which scale as $\sim 1/L^k$ (with integer k), i.e. the powerlike corrections. However they miss the exponential corrections of the kind $\sim e^{-mL}$ (where m is the particle's mass) which physically correspond to the effects of virtual particles propagating around the circle.

In a more general theory one could have different types of particles, and different types could transform into each other during scattering. In this case the S-matrix would have some matrix structure with indices labeling the incoming and the outgoing particles. Then in (2.4) the product in the r.h.s. would actually be a product of matrices, and should be understood as acting on a wavefunction which also carries indices corresponding to particle types.

Let us first present without derivation how Bethe equations will look like in this case. We will need to introduce some important notation. Let us consider a model with K possible particle flavours, or in other words K particle types. For each particle we should also allow linear combinations of different flavours so its flavour state can be though of as an element of $\mathbb{C}^K$. For n particles, their state which we denote by A is then the element of a tensor product

$$A \in H_1 \otimes H_2 \otimes \cdots \otimes H_n \tag{2.6}$$

with each $H_i \simeq \mathbb{C}^K$. Choosing a usual basis e_i in $\mathbb{C}^K$ we can also write

$$A = \sum_{j_1=1}^{K} \cdots \sum_{j_n=1}^{K} A_{j_1 \ldots j_n} e_{j_1} \otimes \cdots \otimes e_{j_n} \ . \tag{2.7}$$

Each S-matrix is a linear operator acting on the tensor product of two of the spaces H_i,

$$\hat{S}_{ij} \in End(H_i \otimes H_j) \tag{2.8}$$

where we put a hat on S to underline that it now has a matrix structure. This of course matches the notation from the part of this collection focussed on S-matrices [1] where the S-matrix has four indices – two for incoming particles and two for outgoing ones. A typical example that we will discuss in the next section is

$$\hat{S}_{12}(p_1, p_2) = f(p_1, p_2)\hat{I} + g(p_1, p_2)\hat{P}_{12} \tag{2.9}$$

where $\hat{I}$ is the identity operator and $\hat{P}$ is the permutation operator, i.e. $\hat{P}_{12}\ (e_a \otimes e_b) = e_b \otimes e_a$, while f and g are some explicit functions. As another illustration, we can write the unitarity condition as

$$\hat{S}_{12}(p_1, p_2)\hat{S}_{12}(p_2, p_1) = 1 \tag{2.10}$$

and the highly important Yang-Baxter equation as

$$\hat{S}_{12}(p_1,p_2)\hat{S}_{13}(p_1,p_3)\hat{S}_{23}(p_2,p_3) = \hat{S}_{23}(p_2,p_3)\hat{S}_{13}(p_1,p_3)\hat{S}_{12}(p_1,p_2) \tag{2.11}$$

and it is satisfied over the space $H_1 \otimes H_2 \otimes H_3$.

Now we are ready to write the Bethe equations for a theory with several particle types. They are similar to (2.4) but the product of S-matrices now acts on a state $A \in H_1 \otimes H_2 \otimes \cdots \otimes H_n$,

$$\boxed{e^{ip_k L}\hat{S}_{k,k+1}\hat{S}_{k,k+2}\dots\hat{S}_{k,n}\hat{S}_{k,1}\dots\hat{S}_{k,k-1}\,A = A, \quad k=1,\dots,n} \tag{2.12}$$

The energy is again the sum of individual energies.

Our main goal is to understand how to solve this equation. Notice that the ordering in the product is also important since S is a matrix[3]. So, the r.h.s. of (2.12) is an operator acting in the full space $H_1 \otimes H_2 \otimes \cdots \otimes H_n$. To solve the Bethe equations (2.12) we need to find its eigenvalues and eigenvectors. Fortunately, this is possible to do in a very efficient way using the fact that the S-matrix satisfies the Yang-Baxter equation. In the next several sections we will demonstrate how this works in concrete examples.

To finish the discussion, let us outline the derivation of the Bethe equations (2.12). First we will need to write the wavefunction in a more explicit form. This was already discussed to some extent in the article on S-matrices [1] (for more details on this see [11] and the review [10]). To automatically take care of the (anti-)symmetrization for identical particles, let us introduce creation and annihilation operators $a_j,\ a_j^\dagger$, whose index j labels the different particle types. Then we can describe the wavefunction as

$$\Psi_{i_1\dots i_n}(x_1,\dots,x_n) = \langle 0|a_{i_1}(x_1)\dots a_{i_n}(x_n)|\Psi(p_1,\dots,p_n)\rangle \tag{2.13}$$

where we use the product of annihilation operators to extract the part of the wavefunction corresponding to particle flavours $i_1,\dots,i_n$. The state $|\Psi(p_1,\dots,p_n)\rangle$ is defined as

$$\begin{aligned} |\Psi(\{p\})\rangle \;\; = \;\; & \int d^n y \sum_{\mathcal{P}\in S_n} A^{\mathcal{P}}_{j_1\dots j_n}(\{p\}) \left(\prod_{m=1}^{n} e^{ip_{\mathcal{P}_m} y_m}\right) \theta(y_1 \ll \dots \ll y_n) \times \\ & a^\dagger_{j_1}(y_1)\dots a^\dagger_{j_n}(y_n)|0\rangle \end{aligned} \tag{2.14}$$

where $\theta(x)$ is the Heaviside step function[4] and we assume summation over repeated indices. The coefficients A are related to each other as

$$A^{\mathcal{P}'} = \hat{S}_{\mathcal{P}_i,\mathcal{P}_{i+1}} \cdot A^{\mathcal{P}} \tag{2.15}$$

[3]These equations again match well the picture of taking one particle around the circle. E.g. after we scatter the 1st particle through the 2nd one they both change flavours, then the 2nd is untouched and we scatter the 1st (with the flavour now different) through the third, etc. So once we take the first particle around the circle it can change the flavour and also all other particles can change the flavour. The result has to match the initial wavefunction which is exactly the statement in (2.12).

[4]I.e. $\theta(y_1 \ll \dots \ll y_n)$ is equal to 1 when $y_1 \ll \dots \ll y_n$ and is equal to zero otherwise. Let us mention that this expression for the wavefunction is valid only in the regions when the particles are well separated, so it makes sense to consider the condition $y_1 \ll \dots \ll y_n$.

if $\mathcal{P}'$ is obtained from $\mathcal{P}$ by permutation of elements $i, i+1$ corresponding to particles $\mathcal{P}_i, \mathcal{P}_{i+1}$. The flavour indices of the S-matrix in (2.15) are understood to be appropriately contracted with those of $A^{\mathcal{P}}$. As any permutation can be written as a sequence of permutations that affect only two elements, any $A^{\mathcal{P}}$ can be related to $A^{\mathcal{I}}$ (with $\mathcal{I}$ being the identity permutation) via a sequence of multiplication by the S-matrices. The Yang-Baxter equation satisfied by the S-matrix ensures this relation is unambiguous.

As an example, for a state with two particles in a theory with only one particle type, we would get

$$\Psi_{x_1 \ll x_2} = e^{ip_1 x_1 + ip_2 x_2} + S(p_1, p_2) e^{ip_2 x_1 + ip_1 x_2} \tag{2.16}$$

$$\Psi_{x_1 \gg x_2} = e^{ip_2 x_1 + ip_1 x_2} + S(p_1, p_2) e^{ip_1 x_1 + ip_2 x_2} \tag{2.17}$$

Exercise: Derive these equations from (2.14).

Let us now recall that we are considering the theory on a circle where the absolute ordering of particles is meaningless, so the ordering is only important up to cyclic permutations. Let us consider for example the wavefunction for $x_1 \ll \cdots \ll x_n$. If we define $y_1 = x_1 + L$ then since the separation between particles cannot be larger than L we have $x_2 \ll \cdots \ll x_n \ll y_1$. For this ordering we would get from (2.14) a different expression for the wavefunction, but it should coincide with the first one as on a circle x_1 is indistinguishable from $x_1 + L$. This leads to

$$e^{-ip_1 L} A^{\mathcal{I}} = \hat{S}_{1,2} \hat{S}_{1,3} \dots \hat{S}_{1,n} A^{\mathcal{I}} \tag{2.18}$$

and in general (notice that the ordering in the product is important since S is a matrix)

$$e^{-ip_k L} A^{\mathcal{I}} = \hat{S}_{k,k+1} \hat{S}_{k,k+2} \dots \hat{S}_{k,n} \hat{S}_{k,1} \dots \hat{S}_{k,k-1} A^{\mathcal{I}} \tag{2.19}$$

These are precisely the equations (2.12) that were announced above, where A is identified with $A^{\mathcal{I}}$.

In conclusion, the crucial problem is to diagonalize the product of S-matrices in (2.12). In the next section we will describe a general procedure for doing this based on the Yang-Baxter equation, and then we will see how it works for concrete examples.

3 Algebraic Bethe ansatz: building the transfer matrix

The method we are going to use for solving the periodicity condition (2.12) goes under the name of the algebraic Bethe ansatz. In this section we will discuss its part which is common for all models – the construction of the so-called transfer matrix – and later we will specialize to concrete examples.

The key insight which allows to diagonalize the product of S-matrices in (2.12) is to introduce an *unphysical* particle with momentum p in an auxiliary space $H_a \simeq \mathbb{C}^K$ and scatter it through all our particles. That is, we define the monodromy matrix

$$\hat{T}_a(p) = \hat{S}_{a1}(p, p_1) \hat{S}_{a2}(p, p_2) \dots \hat{S}_{an}(p, p_n) \tag{3.1}$$

which acts in $H_a \otimes H$ where $H = H_1 \otimes \cdots \otimes H_n$ is our physical Hilbert space. From the Yang-Baxter equation for the S-matrix it follows that the monodromy matrix satisfies a similar condition:

$$\hat{S}_{ab}(p,p')\hat{T}_a(p)\hat{T}_b(p') = \hat{T}_b(p')\hat{T}_a(p)\hat{S}_{ab}(p,p') \tag{3.2}$$

Then we define the transfer matrix by taking a trace over the auxiliary space

$$\hat{T}(p) = \mathrm{Tr}_a \hat{T}_a(p), \tag{3.3}$$

and it is now an operator on the physical space only,

$$\hat{T}(p) \in End(H_1 \otimes \cdots \otimes H_n) \ . \tag{3.4}$$

Remarkably, the transfer matrices for different values of p commute,

$$[\hat{T}(p), \hat{T}(p')] = 0 \quad . \tag{3.5}$$

This follows from the "RTT relation" (3.2)[5] and is the main point of the construction[6]. This commutativity means that they have a common set of eigenvectors.

Moreover, $\hat{T}(p)$ is also related to the product of S-matrices that we want to diagonalize. To show this we need an extra property

$$\hat{S}_{12}(p,p) = -\hat{P}_{12} \tag{3.6}$$

which holds in many theories including all examples we consider below. Then

$$\begin{aligned} \hat{T}(p_1) &= -\mathrm{Tr}_a \hat{P}_{a1}\hat{S}_{a2}(p_1,p_2)\ldots\hat{S}_{an}(p_1,p_n) \\ &= -\mathrm{Tr}_a \hat{S}_{12}(p_1,p_2)\ldots\hat{S}_{1n}(p_1,p_n)\hat{P}_{a1} \\ &= -\hat{S}_{12}(p_1,p_2)\ldots\hat{S}_{1n}(p_1,p_n) \end{aligned} \tag{3.7}$$

and the result is exactly the operator in the r.h.s. of the periodicity condition. We have used

$$\hat{P}_{a1}\hat{S}_{ai} = \hat{S}_{1i}\hat{P}_{a1}, \quad \mathrm{Tr}_a \hat{P}_{ab} = \hat{I}_b \tag{3.8}$$

$$\hat{P}_{ab}^2 = 1 \tag{3.9}$$

$$\hat{P}_{ab} = \hat{P}_{ba} \tag{3.10}$$

$$\hat{P}_{a1}\hat{P}_{a2} = \hat{P}_{a2}\hat{P}_{12} = \hat{P}_{12}\hat{P}_{a1} \tag{3.11}$$

Using cyclicity of the trace we can also show that for any k the transfer matrix $\hat{T}(p_k)$ gives the operator that we want to diagonalize,

$$\hat{T}(p_k) = -\hat{S}_{k,k+1}\hat{S}_{k,k+2}\ldots\hat{S}_{k,n}\hat{S}_{k,1}\ldots\hat{S}_{k,k-1} \tag{3.12}$$

[5]the name 'RTT relation' is due to the fact that in the literature usually one has the R-matrix in place of the S-matrix

[6]Here is one way to explain the construction informally. We want to build a commuting set of operators on the Hilbert space H, and for this we uplift it to $H \otimes H_a \otimes H_b$, then on that space we have $\hat{T}_a$ and $\hat{T}_b$ which 'almost commute' – up to multiplication by S-matrices as in (3.2). Then the operators obtained from $\hat{T}_a, \hat{T}_b$ by tracing over the auxiliary space will really commute with each other on the physical Hilbert space.

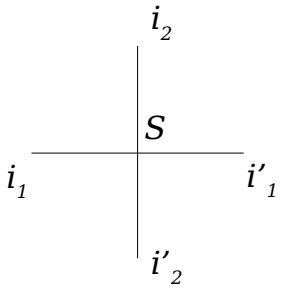

Figure 2: Graphical representation of the S-matrix. The S-matrix $\hat{S}_{12}$ acting in $H_1 \otimes H_2$ is shown as an intersection of two lines. Each of the two lines corresponds to one of the two spaces H_1, H_2. The four ends of the lines correspond to the four indices of the S-matrix.

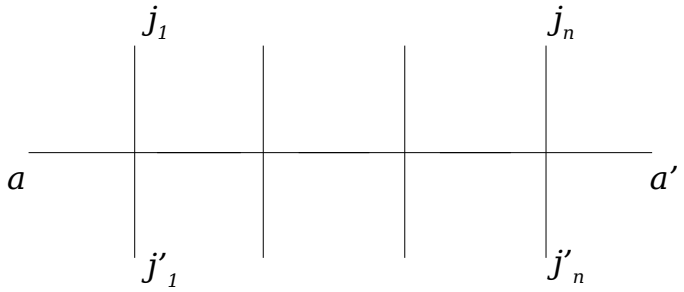

Figure 3: Graphical representation of the monodromy matrix $\hat{T}_a$. The monodromy matrix is a product of several S-matrices. The horizontal line corresponds to the auxiliary space H_a, while vertical lines are associated with the physical spaces $H_1, \ldots, H_n$.

So our goal is to solve the eigenvalue problem for $\hat{T}(p)$,

$$\hat{T}(p)A = \Lambda(p)A \tag{3.13}$$

and then the periodicity condition (2.12) reduces to just an algebraic equation,

$$e^{-ip_k L} = -\Lambda(p_k) \quad ! \tag{3.14}$$

In the next sections we will attack the problem of diagonalizing the transfer matrix $\hat{T}(p)$ for several models.

Let us mention that a pictorial representation is often used for the transfer matrix and the S-matrix, as already discussed in part in other chapters of this collection [12, 1]. To understand how it works, let us write the S-matrix $\hat{S}_{12}$ in index notation. As it acts on the tensor product of two spaces $H_1 \otimes H_2$, its index structure is $S^{i_1 i_2}_{i'_1 i'_2}$ where the first upper index and the first lower index correspond to the H_1 space and the second pair of indices corresponds to the H_2 space. As shown on Fig. 2, we can represent this structure a pair of intersecting lines, with the ends of one line corresponding to the H_1 space and the ends of

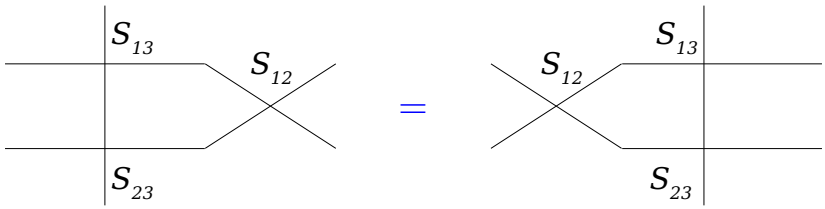

Figure 4: Graphical representation of the Yang-Baxter equation (2.11). The equation means that we can move the vertical line across the intersection point of the two other lines.

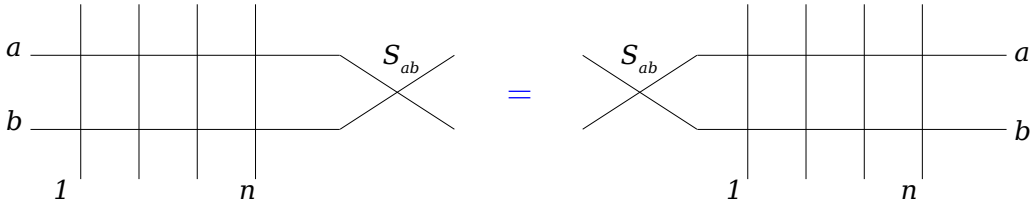

Figure 5: The RTT relation (3.2). The two horizontal lines correspond to auxiliary spaces H_a, H_b, while the vertical lines correspond to physical spaces $H_1, \ldots, H_n$ (labels on the picture next to the lines show which spaces the lines are associated with). Due to the Yang-Baxter equation we can move all the vertical lines one by one to the other side of the intersection, leading to the RTT relation (3.2).

the other line to the H_2 space. The convenience of this notation is that contraction of indices in any expression with a product of operators such as (3.1) is simply represented as joining the corresponding lines. For example the monodromy matrix can be depicted as shown on Fig. 3. The Yang-Baxter equation in graphical form is shown on Fig. 4. Notice that using the pictorial representation one can easily prove the RTT relation using the Yang-Baxter equation, as shown on Fig. 5.

4 Algebraic Bethe ansatz: solving the SU(2) chiral Gross-Neveu model

We will now specialize to a concrete example: the chiral $SU(2)$ Gross-Neveu model (in section 6 we will study the more complicated $SU(3)$ case). This theory has already been discussed and introduced in the part of this collection about S-matrices [1]. We will parameterize the particles' energy and momentum[7] in terms of the rapidity u,

$$E = m\cosh\frac{\pi u}{2}, \quad p = m\sinh\frac{\pi u}{2} \tag{4.1}$$

In the $SU(2)$ chiral Gross-Neveu model there are effectively two massive particles, so in terms of the previous notation $K = 2$ and the S-matrix acts in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Explicitly, the S-matrix reads

$$\hat{S}_{12}(p_1, p_2) = S^{ff}(u_1 - u_2)\hat{R}_{12}^{-1}(u_1 - u_2) \tag{4.2}$$

$$S^{ff}(u) = -\frac{\Gamma(1 - \frac{u}{4i})\Gamma(\frac{1}{2} + \frac{u}{4i})}{\Gamma(1 + \frac{u}{4i})\Gamma(\frac{1}{2} - \frac{u}{4i})} \tag{4.3}$$

where

$$\hat{R}_{12}(u) = \frac{1}{u + 2i}\left(u\hat{I} + 2i\hat{P}_{12}\right) . \tag{4.4}$$

This is the usual R-matrix of the rational type, satisfying the Yang-Baxter equation

$$\hat{R}_{12}(u - v)\hat{R}_{1a}(u)\hat{R}_{2a}(v) = \hat{R}_{2a}(v)\hat{R}_{1a}(u)\hat{R}_{12}(u - v) \tag{4.5}$$

as well as

$$\hat{R}_{12}(u)\hat{R}_{12}(-u) = 1 . \tag{4.6}$$

As we discussed we need to diagonalize the transfer matrix built as a product of these S-matrices with a trace over the auxiliary space. For that the important thing is the matrix structure, so let us drop for some time the prefactor $S^{ff}(u)$, and then

$$\hat{T}(u) = \mathrm{Tr}_a \prod_{i=1}^{N_f} \hat{R}_{ai}^{-1}(u - u_i) = \mathrm{Tr}_a \prod_{i=1}^{N_f} \hat{R}_{ai}(u_i - u) \tag{4.7}$$

where N_f is the number of particles on the circle.

[7] here we choose an unconventional prefactor in front of u for a better match with the usual spin chain notation

For some time now we will concentrate on the problem of diagonalizing this transfer matrix. Later we will return to the Gross-Neveu model and assembling all the ingredients we will write equations for its spectrum.

As we saw, the operators $\hat{T}(u)$ commute for different values of u. As the transfer matrix in our case is a polynomial, its operator coefficients in front of the powers of u all commute with each other. In particular, if we set all u_i to zero, a particular combination of these operators gives the famous XXX spin chain Hamiltonian – a system of N_f interacting spins $s = 1/2$. It is defined as

$$\hat{H} = -\frac{1}{4}\sum_{i=1}^{N_f}(\vec{\sigma}_i\vec{\sigma}_{i+1} - 1) \tag{4.8}$$

where σ_i are the Pauli matrices acting on the i-th site and we identify $i = N_f + 1$ and $i = 1$. The same operator is expressed via the transfer matrix as

$$\hat{H} = i\frac{d}{du}\log\hat{T}(u)\Big|_{u=0} . \tag{4.9}$$

The Hamiltonian can be also written in terms of permutation operators,

$$\hat{H} = \frac{1}{2}\sum_{k=1}^{N_f}\left(1 - \hat{P}_{k,k+1}\right) . \tag{4.10}$$

The importance of this Hamiltonian was first recognized in condensed matter applications where it serves as a model for a ferromagnetic material. In the AdS/CFT context the XXX Hamiltonian is also directly relevant as it describes the leading order anomalous dimensions for operators in a simple subsector of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (see the review [13]).

If we keep u_i nonzero, the relation (4.9) gives the Hamiltonian for the spin chain for which u_i are called the inhomogenieties. The fact that all coefficients of $\hat{T}(u)$ commute with the Hamiltonian, i.e. represent a large number of conservation laws, is a strong sign for the integrability of this model and ultimatley leads to its solution.

To construct the eigenstates of $\hat{T}(u)$ we will use operators originating from the monodromy matrix $\hat{T}_a(u)$. We can write the monodromy matrix explicitly as a 2×2 matrix in auxiliary space,

$$\hat{T}_a(u) = \begin{pmatrix} \hat{A}(u) & \hat{B}(u) \\ \hat{C}(u) & \hat{D}(u) \end{pmatrix} \tag{4.11}$$

where the entries act on the physical space,

$$\hat{A}(u), \hat{B}(u), \hat{C}(u), \hat{D}(u) \in End(H_1 \otimes \cdots \otimes H_n). \tag{4.12}$$

In this notation we have

$$\hat{T}(u) = \hat{A}(u) + \hat{D}(u) . \tag{4.13}$$

The entries of $\hat{T}(u)$ satisfy important commutation relations following from the identity (3.2) which takes the form[8]

$$\hat{R}_{12}(v-u)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)\hat{R}_{12}(v-u)\ . \tag{4.14}$$

In particular,

$$[\hat{B}(u), \hat{B}(v)] = 0 \tag{4.15}$$

$$\hat{A}(v)\hat{B}(w) = \frac{v-w+2i}{v-w}\hat{B}(w)\hat{A}(v) - \frac{2i}{v-w}\hat{B}(v)\hat{A}(w) \tag{4.16}$$

$$\hat{D}(v)\hat{B}(w) = \frac{w-v+2i}{w-v}\hat{B}(w)\hat{D}(v) + \frac{2i}{v-w}\hat{B}(v)\hat{D}(w) \tag{4.17}$$

Exercise: Derive these relations.

Let us introduce the vacuum state $|0\rangle$, in which all spins are up[9]:

$$|0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\ldots\begin{pmatrix}1\\0\end{pmatrix} \tag{4.18}$$

Using the explicit form of the R-matrix we find that the vacuum is an eigenstate of $\hat{T}$ and

$$\hat{A}(u)|0\rangle = |0\rangle,\quad \hat{D}(u)|0\rangle = \prod_j \frac{u_j-u}{u_j-u+2i}|0\rangle,\quad \hat{C}(u)|0\rangle = 0\ . \tag{4.19}$$

The idea is to view $\hat{B}$ as a creation operator and build the transfer matrix eigenstates as

$$|w_1,\ldots,w_{N_a}\rangle = \hat{B}(w_1)\hat{B}(w_2)\ldots\hat{B}(w_{N_a})|0\rangle \tag{4.20}$$

where $w_1, w_2, \ldots$ are parameters known as Bethe roots. Let's see how $\hat{T} = \hat{A} + \hat{D}$ acts on this state. If the second term in the r.h.s. of (4.16) was absent, we could just commute $\hat{A}(u)$ through all $\hat{B}$'s until it hits the vacuum which is its eigenstate. Similarly if there was no second term in the r.h.s. of (4.17) we could commute $\hat{D}(u)$ through all $\hat{B}$'s and then again arrive at the vacuum. However due to the presence of these extra terms in (4.16), (4.17) we will find extra *unwanted* contributions, and the full result is

$$\begin{aligned}\hat{A}(u)|w_1,\ldots,w_{N_a}\rangle &= \prod_j \frac{u-w_j+2i}{u-w_j}\hat{B}(w_1)\ldots\hat{B}(w_{N_a})|0\rangle \\ &+ \sum_j M_j\hat{B}(u)\hat{B}(w_1)\ldots\hat{B}(w_{j-1})\hat{B}(w_{j+1})\ldots\hat{B}(w_{N_a})|0\rangle\end{aligned} \tag{4.21}$$

$$\begin{aligned}\hat{D}(u)|w_1,\ldots,w_{N_a}\rangle &= \prod_j \frac{w_j-u+2i}{w_j-u}\prod_k \frac{u_k-u}{u_k-u+2i}\hat{B}(w_1)\ldots\hat{B}(w_{N_a})|0\rangle \\ &+ \sum_j \tilde{M}_j\hat{B}(u)\hat{B}(w_1)\ldots\hat{B}(w_{j-1})\hat{B}(w_{j+1})\ldots\hat{B}(w_{N_a})|0\rangle\end{aligned} \tag{4.22}$$

[8] the R-matrix has argument $v-u$ as $\hat{T}_a$ is built from $\hat{R}^{-1}$

[9] Sometimes this state is called 'pseudovacuum' rather than vacuum, since e.g. the ground state in this model is actually degenerate, for instance the state with all spins down has the same energy.

The unwanted terms in these two equations are those that include $M_j, \tilde{M}_j$. Explicitly, by M_j we denote the coefficient of the term where A was commuted with B using the second term in the commutation relation (4.16), i.e. the one which exchanges the arguments (and similarly for $\tilde{M}_j$). For instance, it is easy to compute

$$M_1 = \frac{-2i}{u-w_1}\prod_{k\neq 1}\frac{w_1-w_k+2i}{w_1-w_k} \tag{4.23}$$

$$\tilde{M}_1 = \frac{2i}{u-w_1}\prod_{m}\frac{u_m-w_1}{u_m-w_1+2i}\prod_{k\neq 1}\frac{w_k-w_1+2i}{w_k-w_1} \tag{4.24}$$

And since all B's commute, $M_j, \tilde{M}_j$ are trivial generalizations of these expressions,

$$M_j = \frac{-2i}{u-w_j}\prod_{k\neq j}\frac{w_j-w_k+2i}{w_j-w_k} \tag{4.25}$$

$$\tilde{M}_j = \frac{2i}{u-w_j}\prod_{m}\frac{u_m-w_j}{u_m-w_j+2i}\prod_{k\neq j}\frac{w_k-w_j+2i}{w_k-w_j} \tag{4.26}$$

We see that for any j we can cancel the unwanted terms against each other! This will happen if

$$\prod_{k\neq j}\frac{w_j-w_k+2i}{w_j-w_k} = \prod_{m}\frac{u_m-w_j}{u_m-w_j+2i}\prod_{k\neq j}\frac{w_k-w_j+2i}{w_k-w_j} \tag{4.27}$$

It's convenient to relabel the Bethe roots as $\tilde{w}_k = w_k + i$, then dropping the tilde we get

$$\prod_{m}\frac{w_j-u_m+i}{w_j-u_m-i} = \prod_{k\neq j}\frac{w_j-w_k+2i}{w_j-w_k-2i}\ . \tag{4.28}$$

The equations (4.28) are known as Bethe equations for the XXX chain, and they are one of the key results of this section. The eigenvalue of the transfer matrix then reads

$$\Lambda_{SU(2)}(u) = \prod_{m}\frac{u-w_m+i}{u-w_m-i} + \prod_{k}\frac{u-u_k}{u-u_k-2i}\prod_{m}\frac{u-w_m-3i}{u-w_m-i} \tag{4.29}$$

This is the main outcome of our discussion. In particular, we can extract from it the eigenvalue of the XXX Hamiltonian (4.8). To do this we use the relation (4.9) which links the Hamiltonian to the transfer matrix in which all u_i should be set to zero. We find the simple result[10]

$$E = 2\sum_j \frac{1}{w_j^2+1}\ . \tag{4.30}$$

Also, it is important that the XXX spin chain Hamiltonian has an $SU(2)$ symmetry, which will be discussed in more detail in section 5.1, together with its implications for the structure of the eigenstates and eigenvalues.

[10]In the spin chain literature the Bethe roots w_k and the inhomogenieties u_m are usually rescaled by a factor of two compared to our notation so that the Bethe equations would read $\prod_m \frac{w_j-u_m+i/2}{w_j-u_m-i/2} = \prod_{k\neq j}\frac{w_j-w_k+i}{w_j-w_k-i}$ and the energy of the XXX Hamiltonian would be $E = \frac{1}{2}\sum_j \frac{1}{w_j^2+1/4}$

Let us finally underline that the Bethe equations we have just obtained give the spectrum of the transfer matrix and of the spin chain Hamiltonian *exactly* at any length of the chain, i.e. any N_f. This is in contrast with the Bethe ansatz for the 2d field theory we started with, which captures only powerlike and not exponential corrections in the volume L.

One can ask whether these equations provide *all* eigenstates and eigenvalues of the transfer matrix, i.e. whether the algebraic Bethe ansatz solution is complete. While the answer is certainly expected to be positive, a fully rigorous proof has not been found so far (the proofs which are available rely on some conjectures, see [5] for an initial discussion and also [14] as well as references therein for a more recent summary). A related issue is that the Bethe ansatz could have some singular solutions which do not correspond to eigenstates. This issue as well as the question of completeness become more tractable if one introduces twisted boundary conditions for the spinj chain (see e.g. the recent discussion in [15] and references therein). It is also expected that it is sufficient to consider only solutions where the Bethe roots are pairwise distinct in order to get all eigenvalues of the Hamiltonian.

Let us mention that there is a shortcut to the Bethe equations for our transfer matrix. Suppose we forget about the unwanted terms in the commutation relations, then we would still arrive at the same expression for the eigenvalue (4.29). This eigenvalue however appears to have poles when $u = w_j$. The poles cannot be really there as the transfer matrix is not singular at these points[11]. Demanding that the residue of the poles vanish we obtain equations on the roots w_k – which are nothing but the Bethe ansatz equations (4.28) ! This is not a rigorous derivation of the Bethe equations, but this trick is very useful. We will apply it in section 6 for the $SU(3)$ case.

Notice that if we had only the XXX Hamiltonian it would be very hard to guess the transfer matrix and the algebraic Bethe ansatz procedure! Historically the XXX chain was solved first by another method which we will discuss in the next section.

4.1 Coordinate Bethe ansatz for the XXX Hamiltonian

The exact solution of the XXX chain was originally obtained by a more intuitive method known as the coordinate Bethe ansatz [4] (see e.g. [5] for a review and more details on this model). Let us forget about the transfer matrix and consider just the XXX chain Hamiltonian,[12]

$$\hat{H} = \frac{1}{2}\sum_{k=1}^{L}\left(1 - \hat{P}_{k,k+1}\right) \tag{4.31}$$

where as usual we identify the $(L+1)$-th and the 1st sites (we consider the case when there are no inhomogenieties, $u_i = 0$). The method involves making a clever guess (*ansatz*) for the explicit form of the eigenstates. We start with the ground state in which all spins are up,

$$\hat{H}|\uparrow \ldots \uparrow\rangle = 0 \ . \tag{4.32}$$

[11] it's only singular at $u = u_j + 2i$ which corresponds to the pole in our R-matrix

[12] In this subsection as well as in section 5 below we denote the length of the chain as L (to simplify notation) rather than as N_f which was used in the discussion above.

Let us look for the first excited state as a combination of terms

$$|n\rangle = |\uparrow\uparrow \ldots \uparrow\downarrow\uparrow \ldots \uparrow\rangle \tag{4.33}$$

where one spin at the nth site is flipped. Writing

$$|\Psi\rangle = \sum_n e^{ipn}|n\rangle \ , \tag{4.34}$$

and noting also the periodic boundary conditions $n \sim n+L$ we find that it's an eigenstate if

$$e^{ipL} = 1 \ . \tag{4.35}$$

The corresponding energy is

$$E(w) = \frac{2}{w^2+1} \ , \tag{4.36}$$

where we parameterize the momentum as

$$e^{ip} = \frac{w+i}{w-i} \ . \tag{4.37}$$

So it is natural to understand $|\psi\rangle$ as a 1-particle state, and the momentum of the particle is quantized according to (4.35). Also, notice that the energy (4.36) matches the general formula (4.30) for the case with only one root w_j.

Let us further write a two-particle state as

$$|\Psi\rangle = \sum_{1\le n<m\le L} \psi(n,m)|n,m\rangle \ , \tag{4.38}$$

where $|n,m\rangle$ is the state with nth and mth spins flipped. We make an ansatz for the wavefunction ψ as

$$\psi(n,m) = e^{ip_1 n + ip_2 m} + S(p_1,p_2)e^{ip_1 m + ip_2 n} \ , \tag{4.39}$$

where the coefficient S is to be understood as a phase acquired by the wavefunction when the two particles scatter through each other. We find that this will be an eigenstate for

$$S(p_1,p_2) = \frac{w_1 - w_2 + 2i}{w_1 - w_2 - 2i} \ , \tag{4.40}$$

with the eigenvalue $E(u_1) + E(u_2)$. What is truly remarkable is that this construction still works for more than two particles. E.g. for three excitations, denoting

$$|p_1,p_2,p_3\rangle = \sum_{1\le n_1<n_2<n_3\le L} e^{ip_1 n_1 + ip_2 n_2 + ip_3 n_3}|n_1,n_2,n_3\rangle \ , \tag{4.41}$$

we can write the wavefunction as

$$\begin{aligned} |\psi\rangle &= |p_1,p_2,p_3\rangle + S_{12}|p_2,p_1,p_3\rangle + S_{23}|p_1,p_3,p_2\rangle + S_{13}S_{12}|p_2,p_3,p_1\rangle \\ &+ S_{13}S_{23}|p_3,p_1,p_2\rangle + S_{12}S_{13}S_{23}|p_3,p_2,p_1\rangle \end{aligned} \tag{4.42}$$

(with $S_{ij} = S(p_i, p_j)$) and it is still an eigenstate provided the Bethe equations

$$\left(\frac{w_j + i}{w_j - i}\right)^L = \prod_{k \neq j} \frac{w_j - w_k + 2i}{w_j - w_k - 2i} \tag{4.43}$$

are satisfied[13]. Notice that the wavefunction is built using only the same two-particle scattering S-matrix, so in this sense multiparticle scattering is reduced to only $2 \to 2$ interactions. This is in full analogy with the factorization of scattering in 2d integrable QFTs. This spin chain wavefunction is in fact one of the inspirations for writing the QFT wavefunction in the form that we did before.

4.2 Bethe equations for the $SU(2)$ chiral Gross-Neveu model

Let us get back to the $SU(2)$ Gross-Neveu model. Now we are ready to write the full set of Bethe equations for its spectrum. To do this we should plug the explicit expression (4.29) for the transfer matrix eigenvalue Λ into the periodicity condition (3.14). Adding the scalar factor $S^{ff}(u)$ that we dropped before, we get

$$e^{ip_j L} \prod_m S^{ff}(u_j - u_m) \prod_k \frac{u_j - w_k + i}{u_j - w_k - i} = -1 \ . \tag{4.44}$$

These equations should be supplemented by the Bethe equations (4.28) which fix the parameters w_j,

$$\prod_m \frac{w_j - u_m + i}{w_j - u_m - i} = \prod_{k \neq j} \frac{w_j - w_k + 2i}{w_j - w_k - 2i} \ . \tag{4.45}$$

Then the energy is given by

$$E = \sum_j m \cosh \frac{\pi u_j}{2} \ . \tag{4.46}$$

We expect this to be the *exact* result for the energy, up to corrections that are exponentially small in L. This concludes our solution for the spectrum of the Gross-Neveu model at large L.

In the next section we will discuss the XXX chain in some more detail, and then in section 6 derive a generalization of these equations to the $SU(3)$ Gross-Neveu model.

5 Exploring the XXX spin chain

The $SU(2)$ XXX spin chain which we already encountered in the previous section is a very important and widely used model, and deserves a deeper look. In this section we will discuss several of its features in more detail. We will consider the case without any inhomogeneities for clarity.

[13] These equations are of course obtained from (4.28) by setting all u_i to zero.

5.1 The Bethe states in-depth

In the previous section we saw that the XXX spin chain Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{k=1}^{L}\left(1 - \hat{P}_{k,k+1}\right) \tag{5.1}$$

can be diagonalized via algebraic Bethe ansatz and its eigenstates are built as[14]

$$|\Psi\rangle = \hat{B}(w_1)\hat{B}(w_2)\dots\hat{B}(w_M)|\uparrow\uparrow\dots\uparrow\rangle \ , \tag{5.2}$$

where the Bethe roots are determined by

$$\left(\frac{w_j + i}{w_j - i}\right)^L = \prod_{k\neq j}^{M}\frac{w_j - w_k + 2i}{w_j - w_k - 2i} \ . \tag{5.3}$$

Importantly, this Hamiltonian commutes with the operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ giving the total spin of the system, which are defined as a sum of the individual spins, i.e.

$$[\hat{H}, \hat{S}_x] = [\hat{H}, \hat{S}_y] = [\hat{H}, \hat{S}_z] = 0, \tag{5.4}$$

$$\hat{S}_\alpha = \sum_{i=1}^{L}\hat{S}_\alpha^{(i)}, \quad \alpha = x, y, z \ \ . \tag{5.5}$$

These operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ are the generators of the global $SU(2)$ symmetry algebra under which the Hamiltonian is thus invariant. First, this means that we can choose eigenstates of $\hat{H}$ to be eigenstates for $\hat{S}_z$ as well. Also, if we have an eigenstate we can generate more eigenstates (with the same energy) by acting on it repeatedly with the spin-lowering or raising operators $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$.

Since there is an $SU(2)$ algebra acting at each site of the chain, the whole 2^L-dimensional Hilbert space is a tensor product of L copies of the fundamental representation of $SU(2)$. It can be thus decomposed into a direct sum of irreducible representations (irreps) V_α of the global $SU(2)$ symmetry,

$$\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 = \oplus_\alpha V_\alpha \ . \tag{5.6}$$

Due to the relations (5.4) above, each V_α is an invariant subspace for the Hamiltonian and all states there have the same energy. For instance, when $L = 2$ we have a system of two spin-1/2 particles, so the Hilbert space decomposes into a spin-0 and a spin-1 representation. The full space is spanned by the three states

$$|\uparrow\uparrow\rangle, \ |\downarrow\downarrow\rangle, \ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \tag{5.7}$$

[14]To simplify compared to the discussion of the Gross-Neveu model above, in this section we donte the length of the chain as L and the number of excitations as M.

which form the spin-1 irrep of the global $SU(2)$, together with the state

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \tag{5.8}$$

in the singlet (spin-0) representation. The three states (5.7) all have zero energy like the ground state (which is one of them), while the singlet state has the energy $E = 2$.

In each of the representations V_α there is a highest weight state from which all other states are obtained by acting with the lowering operator $\hat{S}_-$. These other states are known as descendants. In our example above with $L = 2$, the states $|\uparrow\uparrow\rangle$ and $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ are highest weight. In fact, the highest weight states are always precisely the Bethe states of the algebraic Bethe ansatz (5.2).[15]

Exercise. What are the solutions of Bethe equations corresponding to the highest weight states for $L = 2$?

There is also a very handy descripton for the descendants, as the lowering operator is expressed through the operator $\hat{B}(w)$ in the limit $w \to \infty$,

$$\hat{B}(w) = \text{const} \times w^{L-1}\hat{S}_- + \ldots, \quad w \to \infty \ . \tag{5.9}$$

It's easy to see that one can add roots $w_k = \infty$ to any solution of the Bethe ansatz and the Bethe equations will still be satisfied. Thus for finite roots the Bethe state (5.2) is a highest weight state, and if some roots are at infinity it is a descendant. Also, due to

$$[\hat{S}_z, \hat{B}(u)] = -\hat{B}(u) \tag{5.10}$$

the Bethe state (5.2) is always an eigenstate of $\hat{S}_z$, with eigenvalue $L/2 - M$, and we can think of this state as having M spins flipped from $|\uparrow\rangle$ to $|\downarrow\rangle$.

As we said above the Bethe ansatz is expected to give the complete spectrum for this model. Also, all highest weight eigenstates can be constructed as Bethe states (5.2). In fact to get all highest weight states it's enough to consider only $M \leq L/2$ in the Bethe equations (this is clear from the fact that the eigenvalue of $\hat{S}_z$ for a Bethe state is $L/2 - M$).

Let us also mention that the algebraic and coordinate Bethe ansatz solutions both rely on the existence of a simple reference eigenstate ($|0\rangle$ in our case), on top of which the other states are constructed. In some integrable models like the anisotropic XYZ chain such a simple reference state is not known in the generic situation, and one has to use other methods to solve them (see [16] for some recent discussion).

Another curious fact is that the Bethe equations have a *potential*. More precisely there exists a function $F(\{w_j\})$ such that the Bethe equations are obtained from its derivative. To see this let us take the logarithm of the Bethe equations, finding

$$L \log \frac{w_j + i}{w_j - i} = \sum_{k=1, k\neq j}^{J} \frac{w_j - w_k + 2i}{w_j - w_k - 2i} + 2\pi i n_j \ , \tag{5.11}$$

[15] And conversely, the states built as (5.2) with finite w_k are highest weight states.

where the integers $n_j \in \mathbb{Z}$ are known as mode numbers[16]. Then we can write these equations for *all* values of k as a derivative of a single function,[17]

$$\frac{\partial F}{\partial w_k} = 2\pi i n_k, \quad n_k \in \mathbb{Z}, \quad k = 1, \dots, M \tag{5.12}$$

where

$$\begin{aligned} F &= L \sum_{j=1}^{M} [(w_j + i)\log(w_j + i) - (w_j - i)\log(w_j - i)] \\ &+ \sum_{k<j}^{M} [(w_k - w_j - 2i)\log(w_k - w_j - 2i) - (w_k - w_j + 2i)\log(w_k - w_j + 2i)] \ . \end{aligned} \tag{5.13}$$

The function F is known as the Yang-Yang function (its analog first appeared in [17]) and can be generalized to almost any other quantum integrable model. In particular it plays an important role in the relation between $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and integrable systems [18]. Some of its properties are further discussed in the lecture course of this collection devoted to Thermodynamic Bethe Ansatz [2].

Let us finally mention for completeness the celebrated Gaudin formula for the norm of the Bethe states [20, 19]. It can be derived almost solely from the commutation relations between elements of the transfer matrix. The result is written as[18]

$$\langle \Psi | \Psi \rangle = (2i)^M \prod_j \frac{w_j}{w_j + 2i} \prod_{j \neq k} \left(1 + \frac{4}{(w_j - w_k)^2}\right) \det_{m,n} \frac{\partial^2 F}{\partial w_m \partial w_n} \tag{5.14}$$

where the key part is the determinant involving the Yang-Yang function. There is also a generalization of this formula, again in determinant form, for a scalar product of two Bethe states in one of which the u_k are "off-shell", i.e. do not necessarilly satisfy the Bethe equations. Finding a compact extension of that formula to a higher rank (e.g. $SU(3)$) chain is a famous and longstanding open problem.

5.2 Spectral curve and finite-gap equations

Let us now discuss some interesting features which emerge in the *classical* limit of the XXX chain – namely the limit when the number of excitations and the length are very large while their ratio is finite. We will see that in this limit the Bethe equations reduce to a set of discontinuity conditions known as the finite-gap equations which are ubiquitous in classical (rather than quantum) integrable systems. These equations also define a Riemann surface known as the classical spectral curve, which encodes the conserved charges of the system.

[16] Sometimes not n but $-n$ is called the mode number in the literature

[17] One could alternatively include in F an extra term $-2\pi i \sum_k n_k w_k$ with some specific n_k, then the Bethe equations would read just $\partial F/\partial w_k = 0$ but F would explicitly depend on the specific choice of the mode numbers n_k which are in general different for different states.

[18] This formula is valid if the set of Bethe roots is invariant under complex conjugation, i.e. $\{w_j\} = \{w_j\}^*$

For a particular simple example we will show how to solve these equations and compute the energy using only various analyticity constraints. For a more detailed discussion we refer the reader to e.g. [21][19]. Our discussion here complements the description of classical integrable systems in another part of the present collection [22].

It is convenient to use the Bethe equations in logarithmic form (5.11), i.e.

$$L \log \frac{w_j + i}{w_j - i} = \sum_{k=1, k\neq j}^{M} \frac{w_j - w_k + 2i}{w_j - w_k - 2i} + 2\pi i n_j \tag{5.15}$$

In our limit L and M are very large. The roots will scale as $w_j \sim L$, and let us use the rescaled roots defined as

$$w_j = L x_j, \quad x_j \sim 1 \tag{5.16}$$

then we have

$$\frac{1}{x_j} = \frac{2}{L} \sum_{k\neq j} \frac{1}{x_j - x_k} + \pi n_j \ . \tag{5.17}$$

The number of roots is very large and they will get close to each other, so instead of a discrete set one can describe them as a continous distribution. The roots will form several cuts in the complex plane. Let us introduce the density of the roots

$$\rho(x) = \frac{1}{L} \sum_j \delta(x - x_j) \tag{5.18}$$

and the resolvent

$$G(x) = \frac{1}{L} \sum_j \frac{1}{x - x_j} = \int_C \frac{d\xi \rho(\xi)}{x - \xi} \ , \tag{5.19}$$

where C is the union of cuts C_i on which the roots condense, $C = \cup C_i$. Introducing the filling fraction

$$\alpha = M/L \tag{5.20}$$

we have the normalization condition

$$\int_C \rho(\xi) d\xi = \alpha \ , \tag{5.21}$$

so that

$$G(x) = \frac{\alpha}{x} + \ldots, \quad x \to \infty \ . \tag{5.22}$$

In our new variables the Bethe ansatz equations (5.17) can be written as

$$G(x + i0) + G(x - i0) = 2 \fint \frac{d\xi \rho(\xi)}{x - \xi} = \frac{1}{x} - \pi n_j, \qquad x \in C_j \tag{5.23}$$

where the dashed integral sign means that we should take the integral's principal value (there is a singularity at $x = \xi$). These relations are known as finite-gap equations[20]. The resolvent

[19] due to our notation there are various factor of 2 differences with [21]

[20] the word 'finite' refers to the fact that we consider the case where the number of cuts C_i is finite

has branch cuts formed by the Bethe roots and is thus a multivalued function whose Riemann surface is known as the classical spectral curve of the model. We derived it from a limiting case of the Bethe ansatz for the quantum XXX chain, but similar curves arise in various other situations, e.g. in classical integrable systems and in matrix models.

Let us also further restrict the solution by imposing an extra condition

$$\prod_j \frac{w_j + i}{w_j - i} = 1 \tag{5.24}$$

where each term in the product is nothing but e^{ip_j} with p_j being the momentum of a single excitation. This condition (known as the "zero-momentum" requirement) in fact means that the Bethe state is invariant under cyclic shifts of the sites. In our limit we can write it as

$$P \equiv \frac{1}{L} \sum_j \frac{1}{x_j} = \pi m, \quad m \in \mathbb{Z} \ , \tag{5.25}$$

or

$$\int d\xi \frac{\rho(\xi)}{\xi} = \pi m \ . \tag{5.26}$$

Since at small x we have

$$G(x) = -\int_C \frac{d\xi \rho(\xi)}{\xi} - x \int_C \frac{d\xi \rho(\xi)}{\xi^2} + \mathcal{O}(x^2) \ , \tag{5.27}$$

we can equivalently write

$$G(0) = -\pi m \ . \tag{5.28}$$

Let us now consider in detail an example when there is only one cut, whose mode number we denote as n. We will fix the resolvent purely from analyticity constraints. From the definition of the resolvent we see that it is an analytic function with $1/x$ asymptotics at $x \to \infty$ and the only singularity being the branch cut formed by the Bethe roots. The discontinuity on the cut is

$$G(x + i0) - G(x - i0) = -2\pi i \rho(x), \quad x \in C \tag{5.29}$$

Furthermore from (5.23) we see that the values of the resolvent above and below the cut sum up to a meromorphic function, thus the cut is of the square root type. Combining all this we can write

$$G(x) = f(x) + g(x)\sqrt{Q(x)} \tag{5.30}$$

where f, g are meromorphic functions and Q is a polynomial. Since there are only two branch points, Q has degree 2. From (5.23) we get

$$f(x) = \frac{1}{2}\left(\frac{1}{x} - \pi n\right) \ . \tag{5.31}$$

With this $f(x)$ the resolvent $G(x)$ has an apparent singularity at $x = 0$ which can only be compensated by $g(x)$. Recalling the asymptotics of $G(x)$ at large x, the only choice is

$g(x) = c/x$ with some constant c. Let us fix the normalization of $Q(x)$ by choosing its free coefficient to be 1, i.e. $Q(x) = ax^2 + bx + 1$. Then the remaining constants a, b, c are fixed from finiteness of $G(x)$ at $x = 0$ together with (5.22), (5.28). This gives

$$G(x) = \frac{1}{2}\left(\frac{1}{x} - \pi n\right) + \frac{1}{2x}\sqrt{(\pi n x)^2 + (4\pi m - 2\pi n)x + 1} \tag{5.32}$$

and

$$\alpha = m/n, \tag{5.33}$$

so we must have $m < n$.

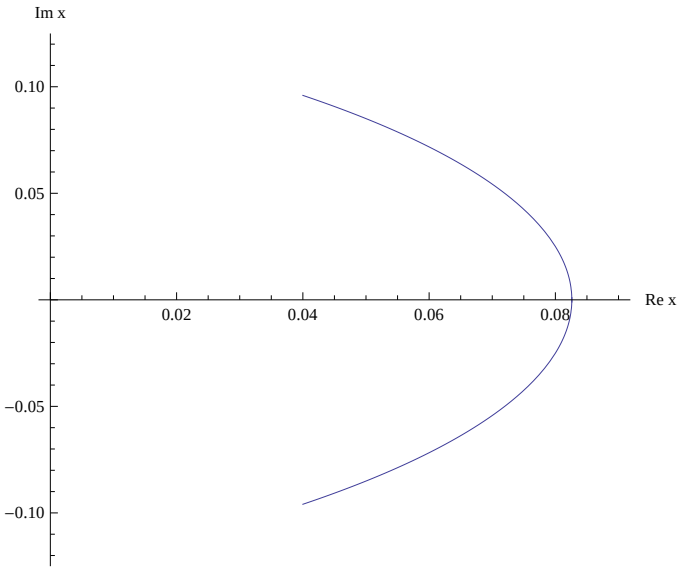

Figure 6: The branch cut of the resolvent (5.32) connecting the two branch points (plot generated from numerical solution of Bethe equations). Notice the bending of the cut.

It's important to understand a subtlety with the choice of branches of the square root in (5.30). Naively one might think that at $x = 0$ the square root in (5.32) will be equal to $\sqrt{1} = +1$, but then the pole at $x = 0$ wouldn't cancel. To clarify this let us consider as an example $n = 3, m = 1$. Then the branch points are at

$$x_1 = \frac{1 + 2i\sqrt{2}}{9\pi}, \quad x_2 = \frac{1 - 2i\sqrt{2}}{9\pi} \tag{5.34}$$

and the cut should connect them. First, strictly speaking the cut is not a straight line in this case. If we take a small part of the cut approximated by the segment $\Delta x \in \mathbb{C}$, then $L\rho(x)\Delta x$ is the number of Bethe roots inside this part of the cut. This quantity should be real, so the condition $\rho(x)dx \in \mathbb{R}$ determines how exactly the cut will bend. Qualitatively it is shown on Fig. 6. Second, far away from the cut, in particular for large positive x, we should have $\sqrt{Q(x)} \sim \pi n x > 0$. However if we start from large positive x and go along the real axis to the point $x = 0$, we will cross the cut. This means that when evaluating $G(x)$ at

$x = 0$ we will have to take $\sqrt{Q(x)} = -1$, not $+1$. Now it's clear that (5.32) indeed satisfies all the constraints we discussed[21].

Finally, to extract the energy, notice that in the classical limit

$$E = 2\sum_j \frac{2}{w_j^2+1} \simeq \frac{1}{L^2}\sum_j \frac{1}{x_j^2} = \frac{2}{L}\int_C \frac{d\xi\rho(\xi)}{\xi^2} \ , \tag{5.35}$$

i.e. the energy is proportional to the linear coefficient of the resolvent's Taylor expansion in (5.27). From (5.32) we thus get

$$E = \frac{2\pi^2 m(n-m)}{L} \ . \tag{5.36}$$

We see that we have found the energy by just imposing the correct analyticity, and not solving the Bethe equations directly at all!

The spectral curve in this case consists of two Riemann sheets joined by a single cut, i.e. it is a sphere. Due to this the solution we found is known as a rational solution. The solution for the case with e.g. two branch cuts, corresponding to a torus, would be called an elliptic solution.

Let us finally note that the finite-gap equations played an important role in the devolpment of integrability in the AdS/CFT context. In particular, the classical limit of Bethe equations derived from gauge theory (conceptually similar to the discussion above) matches the classical spectral curve of the $AdS_5 \times S^5$ integrable string sigma model, providing a nice demonstration of the AdS/CFT duality (see [24] and the review [23]).

6 Bethe ansatz for the SU(3) chiral Gross-Neveu model

Now our goal is to study the $SU(3)$ chiral Gross-Neveu model. There are now more particle types, and we will first discuss only the particles in the fundamental representation of $SU(3)$. That is, we have three particle flavors and the flavor space for each particle is $\mathbb{C}^3$. Our goal is to derive Bethe equations for these excitations. As other particles are their bound states, these Bethe equations are in fact enough to decribe the full spectrum (see e.g. [9]).

6.1 Nested Bethe ansatz for the SU(3) chain

The method we will use is a more advanced version of the algebraic Bethe ansatz from section 4, known as the nested algebraic Bethe ansatz. [22].

[21] In `Wolfram Mathematica` the proper choice of branch cut in the square root $\sqrt{(\pi n x)^2 + (4\pi m - 2\pi n)x + 1}$ appearing in (5.32) would be given for $n = 3, m = 1$ by $3\pi i\sqrt{-i(x-x_1)}\sqrt{-i(x-x_2)}$ (this expression however doesn't take the bending of the cut into account). The extra factors of i ensure that both square root factors have cuts going off to infinity in the same direction, so in their product the cut will disappear except between the branch points.

[22] Let us note that the $SU(3)$ spin chain and many other models with higher rank symmetry group can also be solved by a *coordinate* rather than algebraic version of the nested Bethe ansatz which also proved useful in AdS/CFT (see e.g. [25] and references therien).

It will be more convenient to use a slightly different parameterization of the energy and momentum compared to what we had (Eq. (4.1)) in the $SU(2)$ case, namely we take[23]

$$E = m\cosh\frac{\pi u}{3},\ \ p = m\sinh\frac{\pi u}{3}\ . \tag{6.1}$$

Then extracting from the S-matrix the scalar prefactor (see e.g. [9] and references therein),

$$\hat{S}_{12}(p_1,p_2) = S^{su(3)}(u_1-u_2)\hat{R}_{12}^{-1}(u_1-u_2) \tag{6.2}$$

$$S^{su(3)}(u) = -\frac{\Gamma(1-\frac{u}{6i})\Gamma(\frac{2}{3}+\frac{u}{6i})}{\Gamma(1+\frac{u}{6i})\Gamma(\frac{2}{3}-\frac{u}{6i})}\ , \tag{6.3}$$

we have the R-matrix

$$\hat{R}(u) = \frac{1}{u+2i}(u+2i\hat{P})\ , \tag{6.4}$$

which has the same form as in the $SU(2)$ case but now acts in $\mathbb{C}^3\otimes\mathbb{C}^3$. The main goal is to diagonalize the transfer matrix

$$\hat{T}(u) = \mathrm{Tr}_a\hat{T}_a(u) = \mathrm{Tr}_a\left(\hat{R}_{1a}(u_1-u)\hat{R}_{2a}(u_2-u)\dots\hat{R}_{na}(u_n-u)\right)\ . \tag{6.5}$$

Let us write out its structure in the auxiliary space explicitly, in the following notation:

$$\hat{T}_a(u) = \begin{pmatrix}\hat{T}_{00}(u) & \hat{B}_1(u) & \hat{B}_2(u)\\ \hat{C}_1(u) & \hat{T}_{11}(u) & \hat{T}_{12}(u)\\ \hat{C}_2(u) & \hat{T}_{21}(u) & \hat{T}_{22}(u)\end{pmatrix}\ . \tag{6.6}$$

The commutation relations between the entries follow from the RTT relation as usual. In particular,

$$[\hat{B}_1(u),\hat{B}_1(v)] = 0,\qquad [\hat{B}_2(u),\hat{B}_2(v)] = 0\ , \tag{6.7}$$

$$\hat{B}_1(u)\hat{B}_2(v) = \frac{v-u}{v-u+2i}\hat{B}_2(v)\hat{B}_1(u) + \frac{2i}{v-u+2i}\hat{B}_1(v)\hat{B}_2(u)\ . \tag{6.8}$$

While one can do the calculation in a more abstract way we will use index notation to ensure full clarity. We will have Greek indices $\alpha,\beta,\dots$ and Latin indices $a,b,\dots$, all of which take values 1 and 2. In this notation the relations for commuting the $\hat{T}$'s with the $\hat{B}$'s read

$$\hat{T}_{00}(u)\hat{B}_\alpha(v) = \frac{v-u-2i}{v-u}\hat{B}_\alpha(v)\hat{T}_{00}(u) + \frac{2i}{v-u}\hat{B}_\alpha(u)\hat{T}_{00}(v) \tag{6.9}$$

and

$$\hat{T}_{\alpha\alpha'}(u)\hat{B}_\beta(v) = \frac{v-u+2i}{v-u}\mathbb{R}^{\tau\gamma}_{\alpha'\beta}(v-u)\hat{B}_\gamma(v)\hat{T}_{\alpha\tau}(u) + \frac{2i}{u-v}\hat{B}_{\alpha'}(u)\hat{T}_{\alpha\beta}(v)\ , \tag{6.10}$$

where we marked by red the 'unwanted' terms, i.e. those that later will cancel when we construct the eigenstate and impose the Bethe equations. Remarkably, the $SU(2)$ R-matrix

[23] We use this notation so that the R-matrix in (6.4) that we get here is the same as we had before in the $SU(2)$ case.

which we denote as $\mathbb{R}(u)$, appears in these equations. Explicitly its nonzero elements are, as before,

$$\mathbb{R}^{11}_{11}(u) = \mathbb{R}^{22}_{22}(u) = 1, \tag{6.11}$$

$$\mathbb{R}^{21}_{21}(u) = \mathbb{R}^{12}_{12}(u) = \frac{u}{u+2i}, \qquad \mathbb{R}^{21}_{12}(u) = \mathbb{R}^{12}_{21}(u) = \frac{2i}{u+2i} \tag{6.12}$$

While in the $SU(2)$ case we had $\hat{B}(u)$ as a creation operator, here we have two candidates – $\hat{B}_1(u)$ and $\hat{B}_2(u)$. Let us try to build the eigenvectors as

$$|\Psi\rangle = \sum_{\{a\}} \hat{B}_{a_1}(v_1)\hat{B}_{a_2}(v_2)\dots\hat{B}_{a_n}(v_n)F^{a_1a_2\dots a_n}|0\rangle \tag{6.13}$$

with $a_1, a_2, \dots$ taking the values 1 or 2, and the vacuum is as usual

$$|0\rangle = |\uparrow\uparrow\dots\uparrow\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \dots \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} \ . \tag{6.14}$$

Let's assume for now that there are no unwanted (i.e. red) terms in the commutation relations, and act with $T(u)$ on this state. From T_{00} we get

$$\hat{T}_{00}(u)|0\rangle = |0\rangle \tag{6.15}$$

and

$$\hat{T}_{00}(u)|\Psi\rangle = \prod_k \frac{v_k - u - 2i}{v_k - u}|\Psi\rangle \ . \tag{6.16}$$

Then let's see what we get acting on our state $|\Psi\rangle$ with the rest of the trace of $T(u)$, i.e. with $\sum_\alpha T_{\alpha\alpha}$. Let's take for a start a state $|\Psi\rangle$ with only two excitations,

$$|\Psi\rangle = \hat{B}_{a_1}(v_1)\hat{B}_{a_2}(v_2)F^{a_1a_2}|0\rangle \ . \tag{6.17}$$

Then

$$\begin{aligned}\hat{T}_{\alpha\alpha}(u)|\Psi\rangle &= \frac{v_1 - u + 2i}{v_1 - u}\mathbb{R}^{\tau_1 b_1}_{\alpha a_1}(v_1 - u)\hat{B}_{b_1}(v_1)\hat{T}_{\alpha\tau_1}(u)\hat{B}_{a_2}(v_2)F^{a_1a_2}|0\rangle \qquad (6.18)\\ &= \frac{v_1 - u + 2i}{v_1 - u}\frac{v_2 - u + 2i}{v_2 - u}\mathbb{R}^{\tau_1 b_1}_{\alpha a_1}(v_1 - u)\mathbb{R}^{\tau_2 b_2}_{\tau_1 a_2}(v_2 - u)\hat{B}_{b_1}(v_1)\hat{B}_{b_2}(v_2)\hat{T}_{\alpha\tau_2}(u)F^{a_1a_2}|0\rangle \ .\end{aligned}$$

Finally, the operators $\hat{T}_{\alpha\beta}$ act on the vacuum in a very simple way. That is,

$$\hat{T}_{\alpha\beta}(u)|0\rangle = \delta_{\alpha\beta}\prod_j \frac{u_j - u}{u_j - u + 2i}\ |0\rangle \ . \tag{6.19}$$

So we get

$$\begin{aligned}T_{\alpha\alpha}(u)|\Psi\rangle &= \frac{v_1 - u + 2i}{v_1 - u}\frac{v_2 - u + 2i}{v_2 - u}\prod_j \frac{u_j - u}{u_j - u + 2i} \qquad (6.20)\\ &\times \ \mathbb{R}^{\tau_1 b_1}_{\tau_2 a_1}(v_1 - u)\mathbb{R}^{\tau_2 b_2}_{\tau_1 a_2}(v_2 - u)B_{b_1}(v_1)B_{b_2}(v_2)F^{a_1a_2}|0\rangle \ .\end{aligned}$$

In the indices of the $\mathbb{R}$-matrices there is now a clear pattern, so we see that for any number of excitations as in (6.13) we would get

$$\begin{aligned}\hat{T}_{\alpha\alpha}(u)|\Psi\rangle &= \prod_{k=1}^{n}\frac{v_k-u+2i}{v_k-u}\prod_j\frac{u_j-u}{u_j-u+2i}\prod_{k=1}^{n}\hat{B}_{b_k}(v_k)|0\rangle \\ &\times \mathbb{R}^{\tau_1 b_1}_{\tau_2 a_1}(v_1-u)\mathbb{R}^{\tau_2 b_2}_{\tau_3 a_2}(v_2-u)\ldots\mathbb{R}^{\tau_n b_n}_{\tau_1 a_n}(v_n-u)F^{a_1 a_2\ldots a_n}\ .\end{aligned} \tag{6.21}$$

But this product of $\mathbb{R}$-matrices is the transfer matrix of an $SU(2)$ spin chain! It has free indices $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$, so it acts on the product of vector spaces $V_1\otimes\cdots\otimes V_n$ with each $V_j\simeq\mathbb{C}^2$. And $F^{a_1a_2\ldots a_n}$ is the set of coordinates of a vector in that tensor product, or in other words the wavefunction of an $SU(2)$ chain with n sites. The first upper index and the first lower index of each R-matrix are the indices in the auxiliary space and they are contracted with the neighbouring R-matrices. So we have a product of R-matrices along the auxiliary space and we take a trace over this space. This is precisely the transfer matrix of the $SU(2)$ spin chain on n sites. Notice that $v_1,\ldots,v_n$ are the *inhomogenieties* in this chain.

The a_k indices are contracted with F, so if F is an eigenstate of this transfer matrix we see that $T_{\alpha\alpha}(u)$ will act diagonally on $|\Psi\rangle$ – which is what we want. So we have reduced the initial $SU(3)$ problem to an $SU(2)$ one. This is why the approach we are discussing is called "*nested* algebraic Bethe ansatz".

Of course we already solved the $SU(2)$ spin chain, so we know how to diagonalize this transfer matrix. States will be created by its off-diagonal element, and are parameterized by yet another set of Bethe roots, which we call w_m. (It's the third one – in addition to u_j and v_k.) Then we take as $F^{a_1\ldots a_n}$ the wavefunction of this state:

$$F^{a_1\ldots a_n} = (|\ \text{SU(2) eigenstate}\ \rangle)^{a_1\ldots a_n}\ . \tag{6.22}$$

For example, F^{122} is the coefficient of the term $|\uparrow\downarrow\downarrow\rangle$ in the eigenstate of the $SU(2)$ chain. The new Bethe roots satisfy the $SU(2)$ Bethe equations (4.28) with inhomogenieties set to be v_k:

$$\prod_k\frac{w_m-v_k+i}{w_m-v_k-i} = \prod_{m'\neq m}\frac{w_m-w_{m'}+2i}{w_m-w_{m'}-2i}\ . \tag{6.23}$$

The corresponding eigenvalue of the $SU(2)$ transfer matrix is given by (4.29),

$$\Lambda_{SU(2)}(u) = \prod_m\frac{u-w_m+i}{u-w_m-i}+\prod_k\frac{u-v_k}{u-v_k-2i}\prod_m\frac{u-w_m-3i}{u-w_m-i}\ . \tag{6.24}$$

Then, we can assemble all our calculations to write the result for the full transfer matrix eigenvalue,

$$(\hat{T}_{00}(u)+\hat{T}_{11}(u)+\hat{T}_{22}(u))|0\rangle = \Lambda(u)|0\rangle \tag{6.25}$$

where

$$\Lambda(u) = \prod_k\frac{v_k-u-2i}{v_k-u}+\Lambda_{SU(2)}(u)\prod_k\frac{v_k-u+2i}{v_k-u}\prod_j\frac{u_j-u}{u_j-u+2i}\ . \tag{6.26}$$

During this whole discussion we of course ignored the extra terms in the commutation relations. One can show that they will cancel provided the roots v_k satisfy a set of constraints which are the Bethe equations for this model. Showing this requires some careful and lengthy work, and we will not do this here. Some guidance for a similar model can be found in [26].

Rather than going into details of this derivation we will follow instead a shortcut which gives the same Bethe equations. Recall that for $SU(2)$ one way to derive the Bethe equations is to require cancellation of spurious poles in the transfer matrix eigenvalue. We can do the same here. We see that $\Lambda(u)$ appears to have poles when $u = v_k$ which in fact should be absent as the transfer matrix is not singular at those points. Demanding that the residue at $u = v_{k'}$ vanishes we find

$$0 = (-2i) \prod_{k \neq k'} \frac{v_k - v_{k'} - 2i}{v_k - v_{k'}} + \Lambda_{SU(2)}(v_{k'}) \times 2i \times \prod_{k \neq k'} \frac{v_k - v_{k'} + 2i}{v_k - v_{k'}} \prod_j \frac{u_j - v_{k'}}{u_j - v_{k'} + 2i} \quad (6.27)$$

(note that $\Lambda_{SU(2)}$ is clearly not singular at this point). Plugging in $\Lambda_{SU(2)}(v_{k'})$ from (6.24), where only the first term is nonzero, we get

$$0 = -\prod_{k \neq k'} \frac{v_k - v_{k'} - 2i}{v_k - v_{k'} + 2i} + \prod_m \frac{v_{k'} - w_m + i}{v_{k'} - w_m - i} \prod_j \frac{u_j - v_{k'}}{u_j - v_{k'} + 2i} \ . \quad (6.28)$$

Shifting $v_k = \tilde{v}_k + i, w_m = \tilde{w}_m + i$ and dropping the tildes we get

$$0 = -\prod_{k \neq k'} \frac{v_k - v_{k'} - 2i}{v_k - v_{k'} + 2i} + \prod_m \frac{v_{k'} - w_m + i}{v_{k'} - w_m - i} \prod_j \frac{u_j - v_{k'} - i}{u_j - v_{k'} + i} \quad (6.29)$$

Note that the Bethe equations (6.23) for w's do not change under this shifting. So, we find

$$\prod_j \frac{v_{k'} - u_j + i}{v_{k'} - u_j - i} = \prod_{k \neq k'} \frac{v_{k'} - v_k + 2i}{v_{k'} - v_k - 2i} \prod_m \frac{v_{k'} - w_m - i}{v_{k'} - w_m + i} \ , \quad (6.30)$$

while the w's satisfy the same equations as before in (4.28) but with inhomogeneities v_k,

$$\prod_m \frac{w_j - u_m + i}{w_j - u_m - i} = \prod_{k \neq j} \frac{w_j - w_k + 2i}{w_j - w_k - 2i} \ . \quad (6.31)$$

The equations above determine the eigenvalue of the $SU(3)$ transfer matrix and in particular of the $SU(3)$ version of the XXX spin chain Hamiltonian. Like in the $SU(2)$ case this Hamiltonian is expressed in terms of the transfer matrix as in (4.9)

$$\hat{H} = i \frac{d}{du} \log \hat{T}(u) \Big|_{u=0} \ , \quad (6.32)$$

and can be written explicitly as

$$\hat{H} = \text{const} \times \sum_{k=1}^{N_f} \left(1 - \hat{P}_{k,k+1}\right) \ , \quad (6.33)$$

the only difference with the $SU(2)$ case being that it now acts in the tensor product of $\mathbb{C}^3$ rather than $\mathbb{C}^2$ spaces.

Let us also note that for the $SU(2)$ chain we had one type of Bethe roots (w_k) parameterizing the eigenstates, while here we have two types – v_k and w_k. In fact one should think of each Bethe root type as associated to the nodes on the Dynkin diagram of the symmetry group. In our case, accordingly, $SU(2)$ has a Dynkin diagram with only one node, while for $SU(3)$ there are two nodes. One can also consider spin chains with higher rank symmetry groups G beyond the $SU(2)$ and $SU(3)$ cases. Accordingly, instead of $\mathbb{C}^2$ or $\mathbb{C}^3$ one one would have at each site of the chain a higher-dimensional complex space. At least for most of the compact simple Lie groups G, the corresponding spin chain again can be solved by Bethe ansatz, and the Bethe equations are written in a uniform way in terms of the group's Cartan matrix as well the representation of the group chosen at each site (see e.g. the review [9] and references therein). A similar story continues to hold even for super Lie algebras, in particular for the algebra $\mathfrak{psu}(2,2|4)$ which underlies the structure of Bethe equations describing the spectrum of long operators in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [3].

Finally we can assemble the equations for the spectrum of the $SU(3)$ chiral Gross-Neveu model. Using the expression above for the eigenvalue of the $SU(3)$ transfer matrix, we can write the periodicity condition as

$$e^{ip_j L}\prod_m S^{su(3)}(u_j-u_m)\prod_k \frac{u_j-v_k+i}{u_j-v_k-i}=-1\ , \tag{6.34}$$

together with equations for auxiliary Bethe roots

$$\prod_j \frac{v_{k'}-u_j+i}{v_{k'}-u_j-i}=\prod_{k\neq k'}\frac{v_{k'}-v_k+2i}{v_{k'}-v_k-2i}\prod_m\frac{v_{k'}-w_m-i}{v_{k'}-w_m+i}\ , \tag{6.35}$$

$$\prod_m \frac{w_j-u_m+i}{w_j-u_m-i}=\prod_{k\neq j}\frac{w_j-w_k+2i}{w_j-w_k-2i}\ . \tag{6.36}$$

The energies are as usual a sum of single particle energies (notice we use the parameterization (6.1))

$$E=\sum_j m\cosh\frac{\pi u_j}{3}\ . \tag{6.37}$$

Thus we have completed the solution for the spectrum of the $SU(3)$ Gross-Neveu model in large volume L.

7 Bethe ansatz for the harmonic oscillator

Let us discuss in this last section a completely different setting where Bethe-like equations also appear. Namely, one can use a kind of Bethe ansatz to get eigenstates of the very well-studied one-dimensional harmonic oscillator in quantum mechanics (for a more detailed discussion of this case see e.g. [27]). Thus, we are studying the Schrodinger equation

$$-\frac{\hbar^2}{2m}\psi''(x)+V(x)\psi(x)=E\psi(x) \tag{7.1}$$

with the potential

$$V(x) = \frac{m\omega^2 x^2}{2} \ . \tag{7.2}$$

Let us introduce the so-called quasimomentum

$$p(x) = \frac{\hbar}{i}\frac{\psi'(x)}{\psi(x)} \ , \tag{7.3}$$

in terms of which the Schrodinger equation takes the form

$$p^2 - i\hbar p' = 2m(E - V) \ . \tag{7.4}$$

As $\psi(x)$ is regular, the only singularities of $p(x)$ are at the zeros $x = x_j$ of the wavefunction, where the quasimomentum has simple poles with residue $\frac{\hbar}{i}$.

In the classical limit, i.e. for highly excited states, we get from (7.4)

$$p \simeq p_{cl} = \sqrt{2m(E - V)} \tag{7.5}$$

so now the quasimomentum has a branch cut. This cut can be understood as a collection of poles at x_j, which become denser and denser, eventually forming a smooth distribution giving rise to a cut. The situation is very similar to the classical spectral curve of the XXX model we discussed in section 5.2

Let us now see that for any state (not necessarily semiclassical) we can derive a simple set of equations fixing the positions of these poles. At large x we have

$$p(x) = im\omega x + \mathcal{O}(1/x) \ , \tag{7.6}$$

so we can write

$$p(x) = im\omega x + \frac{\hbar}{i}\sum_{j=1}^{N}\frac{1}{x - x_j} \ . \tag{7.7}$$

From the large x asymptotics of (7.4) we can already find the spectrum! It reads

$$E = \hbar\omega\left(N + \frac{1}{2}\right) \ . \tag{7.8}$$

From (7.4) we also get a set of Bethe-like equations for the roots,

$$x_j = \frac{\hbar}{2\omega m}\sum_{k\neq j}\frac{1}{x_j - x_k} \ . \tag{7.9}$$

They are clearly reminiscent of the usual Bethe ansatz form, with one root in the l.h.s. and interaction between roots in the r.h.s. As one can expect from the usual form of the oscillator wavefunctions, the solutions to this equation are the roots of the N-th Hermite polynomial,

$$H_N\left(\sqrt{\frac{2m\omega}{\hbar}}x_j\right) = 0 \ . \tag{7.10}$$

Knowing the roots x_j we can also reconstruct the wavefunction from (7.3), (7.7). Equations similar to (7.9) frequently arise as limiting cases of the Bethe ansatz equations for other models, e.g. in the limit of large L and fixed number of excitations in the XXX spin chain.

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References

[1] D. Bombardelli, "Lectures on S-matrices and Integrability", arxiv:1605.xxxxx, in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szecsenyi and S. van Tongeren, Journal of Physics A (2016), to appear.

[2] S. van Tongeren, "An integrability primer for the gauge-gravity correspondence: thermodynamic Bethe ansatz", to appear

[3] N. Beisert *et al.*, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982 [hep-th]].

[4] H. Bethe, "On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain," Z. Phys. **71** (1931) 205.

[5] L. D. Faddeev, "How algebraic Bethe ansatz works for integrable model," hep-th/9605187.

[6] B. Sutherland, Beautiful Models: 70 Years of Exactly Solved Quantum Many-Body Problems. World Scientific, 2004.

[7] M. Karbach and G. Müller, "Introduction to the Bethe ansatz I", Computers in Physics 11 (1997) 36, arXiv:cond-mat/9809162.

[8] F. H. L. Essler, H. Frahm, F. Goehmann, A. Kluemper, and V. E. Korepin, The One-Dimensional Hubbard Model, Cambridge University Press (2005).

[9] D. Volin, "Quantum integrability and functional equations: Applications to the spectral problem of AdS/CFT and two-dimensional sigma models," J. Phys. A **44** (2011) 124003 [arXiv:1003.4725 [hep-th]].

[10] S. J. van Tongeren, "Integrability of the $AdS_5 \times S^5$ superstring and its deformations," J. Phys. A **47** (2014) 433001 [arXiv:1310.4854 [hep-th]].

[11] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models," Annals Phys. **120** (1979) 253.

[12] F. Loebbert, "Lectures on Yangian Symmetry", arxiv:1605.xxxxx, in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szecsenyi and S. van Tongeren, Journal of Physics A (2016), to appear.

[13] J. A. Minahan, "Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills," Lett. Math. Phys. **99** (2012) 33 doi:10.1007/s11005-011-0522-9 [arXiv:1012.3983 [hep-th]].

[14] W. Hao, R. I. Nepomechie and A. J. Sommese, "Completeness of solutions of Bethe's equations," Phys. Rev. E **88** (2013) no.5, 052113 doi:10.1103/PhysRevE.88.052113 [arXiv:1308.4645 [math-ph]].

[15] R. I. Nepomechie and C. Wang, "Twisting singular solutions of Bethe's equations," J. Phys. A **47** (2014) no.50, 505004 doi:10.1088/1751-8113/47/50/505004 [arXiv:1409.7382 [math-ph]].

[16] J. Cao, W. L. Yang, S. Cui, K. J. Shi and Y. Wang, "Spin-$\frac{1}{2}$ XYZ model revisit: General solutions via off-diagonal Bethe ansatz," Nucl. Phys. B **886** (2014) 185 doi:10.1016/j.nuclphysb.2014.06.026 [arXiv:1307.0280 [cond-mat.stat-mech]].

[17] C. N. Yang and C. P. Yang, "Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction," J. Math. Phys. **10** (1969) 1115.

[18] N. A. Nekrasov and S. L. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories," arXiv:0908.4052 [hep-th].

[19] V. E. Korepin, "Calculation Of Norms Of Bethe Wave Functions," Commun. Math. Phys. **86** (1982) 391.

[20] M. Gaudin, Journal de Physique 37, 1087 (1976); B. M. McCoy, T. T. Wu, and M. Gaudin, Phys. Rev. D 23, 417 (1981).

[21] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, "Classical/quantum integrability in AdS/CFT," JHEP **0405** (2004) 024 [hep-th/0402207].

[22] A. Torrielli, "Lectures on Classical Integrability", arxiv:1605.xxxxx, in: "An integrability primer for the gauge-gravity correspondence", ed.: A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szecsenyi and S. van Tongeren, Journal of Physics A (2016), to appear.

[23] S. Schafer-Nameki, "Review of AdS/CFT Integrability, Chapter II.4: The Spectral Curve," Lett. Math. Phys. **99** (2012) 169 doi:10.1007/s11005-011-0525-6 [arXiv:1012.3989 [hep-th]].

[24] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, "Complete spectrum of long operators in N=4 SYM at one loop," JHEP **0507** (2005) 030 doi:10.1088/1126-6708/2005/07/030 [hep-th/0503200].

[25] N. Beisert, "The SU(2—2) dynamic S-matrix," Adv. Theor. Math. Phys. **12** (2008) 948 doi:10.4310/ATMP.2008.v12.n5.a1 [hep-th/0511082].

[26] F. H. L. Essler and V. E. Korepin, "Higher conservation laws and algebraic Bethe ansatze for the supersymmetric t-J model," Phys. Rev. B **46** (1992) 9147.

[27] N. Gromov and P. Vieira, "The AdS(5) x S**5 superstring quantum spectrum from the algebraic curve," Nucl. Phys. B **789** (2008) 175 [hep-th/0703191].

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Introduction to the thermodynamic Bethe ansatz

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Abstract

We give a pedagogical introduction to the thermodynamic Bethe ansatz, a method that allows us to describe the thermodynamics of integrable models whose spectrum is found via the (asymptotic) Bethe ansatz. We set the stage by deriving the Fermi-Dirac distribution and associated free energy of free electrons, and then in a similar though technically more complicated fashion treat the thermodynamics of integrable models, focusing on the one dimensional Bose gas with delta function interaction as a clean pedagogical example, secondly the XXX spin chain as an elementary (lattice) model with prototypical complicating features in the form of bound states, and finally the SU(2) chiral Gross-Neveu model as a field theory example. Throughout this discussion we emphasize the central role of particle and hole densities, whose relations determine the model under consideration. We then discuss tricks that allow us to use the same methods to describe the exact spectra of integrable field theories on a circle, in particular the chiral Gross-Neveu model. We moreover discuss the simplification of TBA equations to Y systems, including the transition back to integral equations given sufficient analyticity data, in simple examples.

Contents

1 Introduction

Integrable models are an important class of physical models because they are "solvable" – meaning we can often exactly compute various quantities – while sharing important features with more complicated physical models. In other words, they make great pedagogical tools. Integrability makes it possible to diagonalize the chiral Gross-Neveu model's Hamiltonian for instance [1, 2], giving exact formulas that explicitly demonstrate deep quantum field theoretical concepts such as dimensional transmutation and asymptotic freedom. As part of a series of articles introducing aspects of integrability [3], in this article we describe how integrability is used to describe the exact thermodynamics of integrable models, and relatedly the spectra of integrable field theories defined on a circle, using a method known

as the "thermodynamic Bethe ansatz".

As the name implies, the thermodynamic Bethe ansatz (TBA) revolves around applying the Bethe ansatz in a thermodynamic setting. In essence, the Bethe ansatz description of an integrable model provides us with momenta and energy distributions of particles, which in principle contains the information needed to determine the density of states in the thermodynamic limit, and the associated particle and hole distributions in thermodynamic equilibrium. This approach was pioneered in the late sixties by Yang and Yang [4] who applied it to the Bose gas with delta function interaction, also known as the Lieb-Liniger model [5]. It was quickly adapted to lattice integrable models such as the Heisenberg spin chain [6–8] and Hubbard model [9,10].[1] The TBA can be used to compute the free energy of integrable field theories as well, which upon doing a double Wick rotation has an alternative use in finding their exact ground state energies in finite volume [12]. By a form of analytic continuation excited state energies can also be computed in the TBA approach [13, 14]. These equations can be simplified and reduced to a so-called Y system [15], which is a set of functional relations not limited to a particular state which can be the same for different models. Providing a sufficient amount of analyticity data then singles out a model and state.[2]

In the context of the AdS/CFT correspondence, the worldsheet theory of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string is an integrable field theory, see e.g. [17, 18] for reviews, and its exact energy spectrum can be computed by means of the thermodynamic Bethe ansatz [19–23], as first suggested in [24].[3] This energy spectrum is AdS/CFT dual to the spectrum of scaling dimensions in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). Provided we take the AdS/CFT correspondence to hold rigorously, the thermodynamic Bethe ansatz therefore allows us to find exact two point functions in an interacting, albeit planar, four dimensional quantum field theory, nonperturbatively. From a different point of view, this approach provides high precision tests of the AdS/CFT conjecture. The TBA approach has for instance been successfully matched by explicit field theory results up to five loops for the so-called Konishi operator [30–33]. The TBA can also be used to compute the generalized cusp anomalous dimension (the "quark–anti-quark potential") [34, 35], and for instance extends to the duality between strings on the Lunin-Maldacena background and β deformed SYM [36,37] and the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ string dual to three dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons theory [38, 39]. Though TBA-like equations have not yet made a clear appearance in the computation of three point correlation functions in SYM, we can expect they will do so in the exact solution.

Taking in the above, our motivation for studying the TBA is therefore broadly speaking

[1] While we aim to focus on the basic structure, the TBA and related methods also play an important role in computing more complicated observables such as correlation functions at finite temperature, see e.g. [11].

[2] Going a bit beyond the scope of the present article, such Y systems together with analyticity data can be "reduced" even further via so-called T systems to Q systems. Sometimes we can derive such functional relations by direct computations in a model, which can then be turned into integral equations possibly of TBA type. This comes back in the article by S. Negro [16].

[3] In this context the Y system was conjectured in [25] and the required analyticity data clarified in [26–28]. Reducing this results in a Q system, in this context dubbed the quantum spectral curve [29].

twofold: with it we can describe the thermodynamics of nontrivial interacting models of for instance magnetism and strongly correlated electrons of relevance in condensed matter physics, as well as the exact spectra of integrable field theories that play an important role in for example string theory and the gauge/gravity duality. We will not aim to describe the technical details required for particular applications. Rather, we will focus on the unifying features of the TBA approach, and explain them such that it is clear where and how details of a particular model are to be inserted. We will nevertheless use concrete examples, first of all the original case of the Bose gas as a particularly clean example where the transition from Bethe ansatz to thermodynamic Bethe ansatz is a fairly rigorous derivation. We will also discuss the XXX Heisenberg magnet in the context of spin chains, and the SU(2) chiral Gross-Neveu model in integrable field theory. These models illustrate complicating hypotheses in the TBA approach to general integrable models: the presence of multiple interacting particle species, as well as bound state solutions.

We will begin our discussion with free electrons, a trivially integrable model, where we can link our approach to standard statistical physics. This allows us to introduce the concept of density of states, particle and hole density, and the computation of the associated free energy, and reproduce the well known Fermi-Dirac distribution. Following Yang and Yang's original paper, we then extend this framework to the delta function Bose gas. Continuing to the XXX spin chain and SU(2) chiral Gross-Neveu model in the same spirit, requires us to introduce the so-called string hypothesis, which ultimately results in an infinite set of TBA equations. We discuss how these TBA equations can be "simplified" and reduced to a so-called Y system. Next we discuss the TBA approach to exact ground state energies, and indicate how excited state TBA equations can be obtained by analytic continuation, motivated by a toy model example. Relatedly, we discuss the link between the TBA equations and so-called Lüscher corrections, providing analyticity data for excited states. We briefly discuss universality of the Y system for excited states, how to transfer between TBA and Y system plus analyticity data, and the relation of the analyticity data to specific models and states. Two appendices contain details on integral identities and some comments on numerically solving TBA equations.

2 The thermodynamic Bethe ansatz

In an integrable model we usually have a set of Bethe ansatz equations that determines the momenta of particles of any state of the theory, either exactly, or approximately in a large volume limit. In what follows we will assume these to be given, for instance following the discussion in [40]. Combining these Bethe equations with the dispersion relation of the theory under consideration, we can determine its (approximate) energy spectrum. What if we are interested in the thermodynamic limit? Since we can in principle determine the possible and actual momentum distributions of particles for *any* given set of finite quantum numbers (at large volume), we might be able to determine nontrivial thermodynamic quantities by summing up many contributions. The technical way to do this goes under the name of

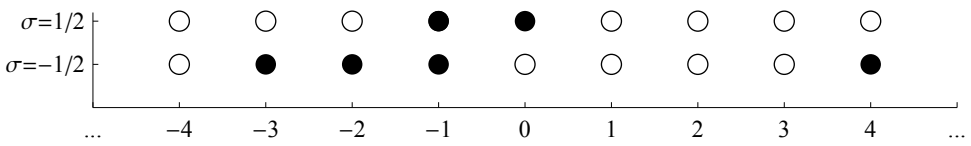


Figure 1: The quantum number lattice for electrons. States of an N free electron state on a circle can be labeled by a set of N integers, split in sets of distinct ones for each spin. Here these integers are represented by filled dots, open dots representing available (unoccupied) quantum states, depicting a state with two spin up electrons and four spin down electrons, with momenta $-2\pi/L$ and 0, and $-6\pi/L, -4\pi/L, -2\pi/L$, and $8\pi/L$ respectively.

the thermodynamic Bethe ansatz, as originally developed by Yang and Yang for the one dimensional Bose gas with delta function interaction potential [4]. We will get to this model and the chiral Gross-Neveu model shortly, but let us begin with a trivially integrable model: free electrons. Our discussion will be similar to section 5.1 of [10].

2.1 Free Fermi gas

Free electrons on a circle are an exactly solvable model. Since the particles do not interact (except for Pauli exclusion), wavefunctions are just superpositions of standing waves on the circle, each coming with a momentum quantization condition

$$e^{ip_jL} = 1 \implies p_j = \frac{2\pi n_j}{L}. \tag{2.1.1}$$

Were we to consider fermions on a periodic lattice (with spacing one), mode numbers would of course only be meaningful modulo L. The Pauli exclusion principle now simply requires that each state is made up of electrons with distinct sets of quantum numbers (including spin). Note that the above equations are nothing but the simplest of Bethe equations. In fact, you might recall that in the Bethe ansatz two identical particles by construction cannot have equal momenta either, which is why we are looking at free fermions rather than free bosons. An N particle state can now be classified by N quantum numbers n_j, split in two sets $\{n_j^\sigma\}$ of distinct numbers, where $\sigma = \pm\frac{1}{2}$ denotes spin of the electrons, cf. figure 1. In this integer space, the number of possible states per unit interval – the total density of states – is one. Due to the linear relation between momentum and these integers, the total density of states for free electrons of spin σ in momentum space is also constant,

$$\rho_\sigma(p_i) \equiv \frac{1}{L}\frac{1}{p_{i+1} - p_i} = \frac{1}{2\pi}. \tag{2.1.2}$$

As usual in thermodynamics we will introduce the partition function

$$Z = \sum_n \langle\psi_n|e^{-\beta H}|\psi_n\rangle = e^{-\beta F}, \tag{2.1.3}$$

where $\beta = 1/T$ is the inverse temperature, and F is the free energy. From here you can compute various thermodynamic quantities, especially upon including chemical potentials

(in H if you wish). In particular, via various paths familiar from basic statistical mechanics, you can derive the momentum distribution of free fermions in thermal equilibrium

$$\rho_{\rm FD}(p) = \frac{1}{2\pi}\frac{1}{1+e^{E(p)/T}}, \tag{2.1.4}$$

known as the Fermi-Dirac distribution. Here $E(p)$ is the dispersion relation of the fermions. We will directly compute the full partition function for free fermions in the large volume limit, in a way that will extend to general integrable models where we only have an implicit description of states at asymptotically large volume.

In the large volume limit, states with finite numbers of particles contribute negligibly to the partition function so we will consider the limit $L \to \infty$ considering states with finite density N_σ/L, N_σ denoting the number of electrons with spin σ. These N_σ particles have distinct momenta that need to occupy N_σ of the allowed values of momentum. If a momentum value is taken we will talk of a particle with this momentum, and if it is not, a hole, as in figure 1. Since we want to describe finite density states, let us introduce densities for particles and holes as

$$L\rho^f_\sigma(p)\Delta p = \#\text{of particles with spin } \sigma \text{ and momentum between } p \text{ and } p+\Delta p,$$
$$L\bar\rho^f_\sigma(p)\Delta p = \#\text{of holes with spin } \sigma \text{ and momentum between } p \text{ and } p+\Delta p.$$

By definition these add up to the total momentum density of states, i.e.

$$\rho^f_\sigma(p) + \bar\rho^f_\sigma(p) = \rho_\sigma(p) = \frac{1}{2\pi}. \tag{2.1.5}$$

Now, to compute the partition function in a thermodynamic picture we need the free energy $F = E - TS$, in other words the energy and entropy of possible configurations. By definition the energy density of any given state is

$$e = \frac{1}{L}\sum_\sigma \sum_{j=1}^{N_\sigma} E_\sigma(p_j), \tag{2.1.6}$$
$$= \sum_j \sum_\sigma E_\sigma(p_j)\frac{p_{j+1}-p_j}{L(p_{j+1}-p_j)}, \tag{2.1.7}$$
$$= \sum_j \sum_\sigma E_\sigma(p_j)(p_{j+1}-p_j)\rho^f_\sigma(p_j), \tag{2.1.8}$$

where the last line is nicely of the form of a discretized integral, appropriate for the large volume limit. There we get

$$e = \int_{-\infty}^{\infty} dp \sum_\sigma E_\sigma(p)\rho^f_\sigma(p), \tag{2.1.9}$$

where we write $\rho^f_\sigma(p)$ for the $L\to\infty$ limit of $\rho^f_\sigma(p_j)$. In a lattice model we would integrate from 0 to 2π (given appropriate normalization choices). Next we want to find an expression for the entropy, the logarithm of the number of available states. By definition

$$\Delta S(p_j) = \log \prod_\sigma \frac{(L\Delta p_j \rho_\sigma(p_j))!}{(L\Delta p_j \rho^f_\sigma(p_j))!(L\Delta p_j \bar\rho^f_\sigma(p_j))!} \tag{2.1.10}$$

which in the large volume limit we can approximate via Stirling's formula, $\log n! = n \log n - n + \mathcal{O}(\log n)$, as

$$\Delta S(p_j) = L\Delta p_j \sum_\sigma \rho_\sigma(p_j) \log \rho_\sigma(p_j) - \rho^f_\sigma(p_j) \log \rho^f_\sigma(p_j) - \bar{\rho}^f_\sigma(p_j) \log \bar{\rho}^f_\sigma(p_j). \tag{2.1.11}$$

In the thermodynamic limit the entropy density is thus given by

$$s = \int_{-\infty}^{\infty} dp \sum_\sigma \rho_\sigma(p) \log \rho_\sigma(p) - \rho^f_\sigma(p) \log \rho^f_\sigma(p) - \bar{\rho}^f_\sigma(p) \log \bar{\rho}^f_\sigma(p). \tag{2.1.12}$$

Putting all this together we find that the free energy density f at temperature T, $f = e - Ts$, is given by

$$f = \int_{-\infty}^{\infty} dp \sum_\sigma E_\sigma(p)\rho^f_\sigma(p) - T(\rho_\sigma(p) \log \rho_\sigma(p) - \rho^f_\sigma(p) \log \rho^f_\sigma(p) - \bar{\rho}^f_\sigma(p) \log \bar{\rho}^f_\sigma(p)). \tag{2.1.13}$$

This is a functional of the densities ρ, and thermodynamic equilibrium corresponds to its stationary point. To find this stationary point we should vary f with respect to ρ^f_σ and $\bar{\rho}^f_\sigma$, but these are not independent! The hole and particle densities are constrained by eqn. (2.1.5), which means

$$\delta\bar{\rho}^f_\sigma = -\delta\rho^f_\sigma. \tag{2.1.14}$$

We then have

$$\delta f = \int_{-\infty}^{\infty} dp \sum_\sigma E_\sigma(p)\delta\rho^f_\sigma(p) - T\left(\log \frac{\rho_\sigma(p)}{\rho^f_\sigma(p)}\delta\rho^f_\sigma(p) + \log \frac{\rho_\sigma(p)}{\bar{\rho}^f_\sigma(p)}\delta\bar{\rho}^f_\sigma(p)\right) \tag{2.1.15}$$

$$= \int_{-\infty}^{\infty} dp\delta\rho^f_\sigma(p)\left(\sum_\sigma E_\sigma(p) - T\log \frac{\bar{\rho}^f_\sigma(p)}{\rho^f_\sigma(p)}\right) = 0, \tag{2.1.16}$$

from which we conclude

$$\frac{\bar{\rho}^f_\sigma(p)}{\rho^f_\sigma(p)} = e^{E_s(p)/T}. \tag{2.1.17}$$

Together with eqn. (2.1.5) this gives

$$\rho^f_\sigma(p) = \frac{1}{2\pi}\frac{1}{1 + e^{E_\sigma(p)/T}}, \tag{2.1.18}$$

which is nothing but the Fermi-Dirac distribution (2.1.4) (here derived in infinite volume). Now we can insert this and the corresponding $\bar{\rho}^f_\sigma$ back into the free energy to find

$$f = -T\int_{-\infty}^{\infty} \frac{dp}{2\pi} \sum_\sigma \log(1 + e^{-E_\sigma(p)/T}). \tag{2.1.19}$$

This is the well known infinite volume free energy of a Fermi gas.

We would like to follow this approach to describe the thermodynamics of general integrable models, where the relation between particle and hole densities is not as simple as eqn. (2.1.5), but nevertheless known. Let us begin with the integrable model for which this was originally done.

2.2 The Bose gas

The Bose gas, also known as the Lieb-Liniger model, is a system of N bosons interacting via a repulsive delta function interaction. The Hamiltonian is given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i>j} \delta(x_i - x_j), \tag{2.2.1}$$

with $c > 0$, and we consider it on a circle of circumference L. This model was 'solved' by Bethe ansatz in [5]. Based on this the thermodynamics of the model were described by Yang and Yang [4], leading to what is now known as the thermodynamic Bethe ansatz. In this section we follow their timeless 1968 paper fairly directly. The nice point about this model is that some things we will have to assume later, can be made precise here. The starting point for our analysis will be the Bethe equations of the Bose gas

$$e^{ip_j L} = \prod_{k \neq j}^{N} \frac{p_j - p_k + ic}{p_j - p_k - ic}, \tag{2.2.2}$$

from which we see that we have an S-matrix given by

$$S(p_l, p_m) = S(p_l - p_m) = \frac{p_l - p_m - ic}{p_l - p_m + ic}. \tag{2.2.3}$$

The solutions of these equations are real.[4] The dispersion relation of these bosons is just the free $E(p) = p^2$.

To get the momentum density of states we need to take a logarithm of the Bethe equations, just as we did for free particles above. To do so we note that

$$S(p) = -e^{2i \arctan p/c} \equiv -e^{i\psi(p)}, \tag{2.2.4}$$

so that we get

$$2\pi I_j = p_j L - i \sum_k \log S(p_j - p_k) = p_j L + \sum_k \left(\psi(p_j - p_k) + \pi\right), \tag{2.2.5}$$

which is all defined up to the integer I_j defining the branch of the logarithm that we take. In the original paper the factor of $N\pi$ is absorbed in these (then possibly half) integers; we simply take the logarithm of the S-matrix on the right hand side, as this naturally generalizes to any model. These integers I_j are in one to one correspondence with solutions of the Bethe equations, just as for the free particle. To prove this, Yang and Yang introduced what is now known as the Yang-Yang–functional.

[4] Consider the equation for the momentum with maximal imaginary part (pick one in case there are multiple), then the right hand side of the equation necessarily has norm greater than or equal to one. The left hand side however has norm less than or equal to one. Therefore we conclude the maximal imaginary part is zero. Similarly, the minimal imaginary part is zero.

2.2.1 The Yang-Yang–functional

Let us define

$$B(p_1,\ldots,p_N) = \tfrac{1}{2}L\sum_{l=1}^{N} p_l^2 - \pi\sum_{j=1}^{N}(2I_j+N-1)p_j + \tfrac{1}{2}\sum_{n,m}\left(\psi_1(p_n-p_m)\right), \tag{2.2.6}$$

where

$$\psi_1(p) = \int_0^p \psi(p')dp' = \int_0^p 2\arctan\tfrac{p'}{c}\,dp'. \tag{2.2.7}$$

The nice thing is that by construction B is an 'action' with the Bethe equations (2.2.2) as 'equations of motion'. Moreover, the matrix $\partial^2 B/\partial k_i \partial k_j$ is positive definite, since the first term in B contributes positively, the second nothing, and the third is positive-semidefinite since $\psi'(p) \geq 0$. So B has a unique extremum, a minimum, whose location is determined by solutions to the Bethe equations. Furthermore, all involved quantities clearly depend continuously on c (via the S matrix). Now in the limit $c \to \infty$ we want to find the wavefunction for N free particles, under the constraint that it vanishes when any two of its arguments coincide, thanks to the infinitely strong repulsion at coincidence. Playing around with this problem a bit in the way that we learn in a course on quantum mechanics, we would find that such wave functions are precisely of Bethe ansatz form, with $S = -1$, precisely the $c \to \infty$ limit of our S matrix. At this point we have

$$p_j = (2I_j + N - 1)\pi/L, \tag{2.2.8}$$

i.e. the momenta are uniquely identified by the integers I (for a given number of particles N). By continuity in c we see that the solutions of the Bethe equations are given by unique sets of distinct momenta in one to one correspondence with sets of distinct integers I, which form a complete set of solutions. We can view these I's as quantum numbers for our problem, just as they were for free electrons.

2.2.2 Thermodynamics

Now we are in a position to apply the ideas of the previous section on free fermions to the Bose gas. To start with, we should understand the relation between the quantum numbers and the momenta in more detail. Let us introduce the so-called counting function $c(p)$ as

$$Lc(p) = \frac{L}{2\pi}p + \frac{1}{2\pi i}\sum_k \log S(p-p_k). \tag{2.2.9}$$

For the Bose gas you can explicitly see that this is a monotonically increasing function. Now, if we have a state with quantum numbers $\{I\}$, by definition the particle momenta correspond to the p's for which $Lc(p_j) = I_j$. By analogy we then say that any allowed quantum number $J \notin \{I\}$ represents a hole with momentum $Lc(p) = J$. We can schematically depicted this situation in figure 2. The corresponding physical picture is as follows. Since each particle carries energy p^2, by monotonicity of the counting function it is clear that the N particle ground state has quantum numbers running between $-\lfloor (N-1)/2 \rfloor$ and $\lfloor (N-1)/2 \rfloor$ (in

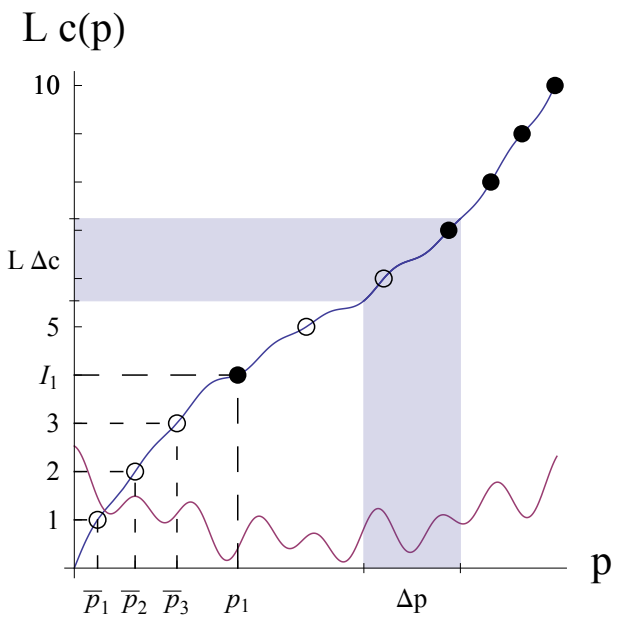


Figure 2: The counting function for a hypothetical distribution of roots. The blue line denotes L times the counting function, which takes integer values at fixed values of momenta, indicated along the function by dots. Open dots indicate unoccupied integers (holes), filled dots particles. For instance the first particle momentum p_1 corresponds to quantum number $Lc(p_1) = 4$. The red line is the (everywhere positive) derivative of the counting function.

unit steps). Excited states now correspond to particles living on the same quantum number lattice (cf. the previous subsection). One or more of them have been moved out of the ground state interval to higher quantum numbers, however, leaving one or multiple 'holes' behind in the ground state lattice, cf. figures 1 and 2.

As before we introduce densities for the particles and holes as

$$L\rho^b(p)\Delta p = \#\text{of particles with momentum between } p \text{ and } p + \Delta p,$$
$$L\bar{\rho}^b(p)\Delta p = \#\text{of holes with momentum between } p \text{ and } p + \Delta p.$$

Again the total density of states in quantum number space is one, which in momentum space picks up a measure factor (Jacobian), cf. figure 2, and we find

$$\rho^b(p) + \bar{\rho}^b(p) = \rho(p) = \frac{dc(p)}{dp}, \tag{2.2.10}$$

where we have replaced the discrete derivative by the continuous one appropriate for the thermodynamic limit, and we keep the normalization by $2\pi/L$ introduced when discussing free electrons. In the Bethe equations we encounter sums over particles, which become integrals over densities since as before

$$\frac{1}{L}\sum_{k\neq j}^{N} \log S(p_j - p_k) = \sum_{k\neq j}^{N} \log S(p_j - p_k)\frac{p_k - p_{k+1}}{L(p_k - p_{k+1})} \to \int_{-\infty}^{\infty} dp' \log S(p_{(j)} - p')\rho^b(p').$$

Using relation (2.2.10) to also express the left hand side of the Bethe equations in terms of

densities we find

$$\rho^b(p) + \bar{\rho}^b(p) = \frac{1}{2\pi} + K \star \rho^b(p), \tag{2.2.11}$$

where

$$K(p) = \frac{1}{2\pi i}\frac{d}{dp}\log S(p), \tag{2.2.12}$$

and $\star$ denotes the convolution[5]

$$f \star g\,(p) \equiv \int_{-\infty}^{\infty} dp' f(p-p')g(p'). \tag{2.2.13}$$

Equation (2.2.11) is the thermodynamic analogue of the Bethe equations, and the analogue of the constraint (2.1.5) for free particles (note that eqn. (2.2.11) actually reduces to (2.1.5) for a trivial S matrix). Now we are in the same position as we were for free electrons.

The free energy is of the same form as before,

$$f = \int_{-\infty}^{\infty} dp \left(E\rho^b - T\left(\rho\log\rho - \rho^b\log\rho^b - \bar{\rho}^b\log\bar{\rho}^b\right)\right), \tag{2.2.14}$$

where we recall that for our almost free bosons $E(p) = p^2$. To describe thermodynamic equilibrium we should now vary f with respect to ρ^b and $\bar{\rho}^b$, subject to eqn. (2.2.11) meaning

$$\delta\bar{\rho}^b = -\delta\rho^b + K \star \delta\rho^b. \tag{2.2.15}$$

The result is a little more complicated than before

$$\delta f = \int_{-\infty}^{\infty} dp \left(E\delta\rho^b - T\left(\log\frac{\rho}{\rho^b}\delta\rho^b + \log\frac{\rho}{\bar{\rho}^b}\delta\bar{\rho}^b\right)\right) \tag{2.2.16}$$

$$= \int_{-\infty}^{\infty} dp\,\delta\rho^b \left(E - T(\log\frac{\bar{\rho}^b}{\rho^b} + \log\left(1 + \frac{\rho^b}{\bar{\rho}^b}\right)\tilde{\star}\,K)\right) \tag{2.2.17}$$

where $\tilde{\star}$ denotes 'convolution' from the right,

$$f\,\tilde{\star}\,K(p) = \int_{-\infty}^{\infty} dp' f(p')K(p'-p). \tag{2.2.18}$$

Introducing the pseudo-energy ϵ by analogy to the free fermion case

$$\frac{\bar{\rho}^b}{\rho^b}(p) = e^{\epsilon(p)/T}, \tag{2.2.19}$$

we see that in thermodynamic equilibrium it needs to satisfy

$$\epsilon(p) = E(p) - T\log(1 + e^{-\epsilon/T})\,\tilde{\star}\,K \tag{2.2.20}$$

known as a thermodynamic Bethe ansatz equation. This equation can be numerically solved by iteration, as clearly discussed in appendix A of the original paper [4]. We briefly discuss

[5] In models where the momenta do not enter the S matrix in difference form, the derivative in K refers to the first argument (p of $S(p,p')$), while the convolution would become an integral over the second (p'). We will only encounter models where we can pick a parametrization that gives a difference form.

some general aspects of solving TBA equations numerically in appendix B. Given a solution of this equation, the free energy in thermodynamic equilibrium is given by

$$f = -T \int_{-\infty}^{\infty} \frac{dp}{2\pi} \log(1 + e^{-\epsilon/T}). \tag{2.2.21}$$

The above formulae are frequently written in terms of a Y function $Y = e^{\epsilon(p)/T}$.

In summary, starting with the Bethe ansatz solution of the one dimensional Bose gas with δ function interaction, we can continue to use concepts like density of states as we did for free electrons, because individual momenta are still conserved. The nontrivial S matrix of the model now results in an integral equation for the particle density in thermodynamic equilibrium. In this way we reduce the computation of the infinite volume partition function of an *interacting* theory to an integral equation that we can solve rather easily at least numerically, for any value of the coupling c.

In a general integrable model the situation is a little more complicated if its excitation spectrum contains bound states of elementary excitations. The XXX spin chain is such a model, and furthermore represents the internals of the chiral Gross-Neveu model.

2.3 The XXX spin chain

The Heisenberg XXX spin chain is a one dimensional lattice model with Hamiltonian

$$H = -\frac{J}{4} \sum_{i=1}^{N_f} (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - 1), \tag{2.3.1}$$

where $\vec{\sigma}$ is the vector of Pauli matrices. We take the lattice to be periodic; $\sigma_{N_f+1} = \sigma_1$. This Hamiltonian acts on a Hilbert space given by N_f copies of $\mathbb{C}^2$, one for each lattice site i. Identifying $(1,0)$ as $|\uparrow\rangle$ and $(0,1)$ as $|\downarrow\rangle$, states in this Hilbert space can be viewed as chains of spins, in this case closed. For $J > 0$ this is a model of a ferromagnet where spins prefer to align, while for $J < 0$ we have an antiferromagnet where spins prefer to alternate.

The Bethe equations for this model are

$$e^{i\mathrm{p}_i N_f} \prod_{j=1}^{N_a} S^{11}(v_i - v_j) = -1, \tag{2.3.2}$$

where

$$\mathrm{p}_i = \mathrm{p}(v_i), \qquad \mathrm{p}(v) = -i \log S^{1f}(v), \tag{2.3.3}$$

and

$$S^{11}(w) = \frac{w - 2i}{w + 2i}, \quad S^{1f}(w) = \frac{w + i}{w - i}. \tag{2.3.4}$$

These equations are the homogeneous limit of the auxiliary Bethe equations of the chiral Gross-Neveu model we will encounter later, where the "f" will stand for the fermions of this model. The reason for the remaining notation will become apparent soon. The energy eigenvalue associated to a solution of these Bethe equations is

$$\mathrm{E} = \sum_i \mathrm{E}_1(v_i), \quad \text{where} \quad \mathrm{E}_1(v) = -2J \frac{1}{v^2 + 1}. \tag{2.3.5}$$

2.3.1 The string hypothesis

To describe the thermodynamics of this model, we would like to understand the type of solutions these equations can have, specifically as we take the system size N_f to infinity.[6] The situation will be considerably different from the Bose gas that we just discussed, because here we can have solutions with complex momenta,[7] For real momenta nothing particular happens in our equations, and we simply get many more possible solutions as N_f grows. If we consider a solution with complex momenta, however, say a state with $\mathrm{Im}(\mathrm{p}_1) > 0$, we have an immediate problem:

$$e^{i\mathrm{p}_q N_f} \to 0, \quad \text{as } N_f \to \infty. \tag{2.3.6}$$

We see that the only way a solution containing p_1 can exist in this limit is if this zero is compensated by a pole in one of the S^{11} (eqn. (2.3.4)), which can be achieved by setting

$$v_2 = v_1 + 2i. \tag{2.3.7}$$

At this point we have fixed up the equation for p_1, but we have introduced potential problems in the equation for p_2. Whether there is a problem can be determined by multiplying the equations for p_1 and p_2 so that the singular contributions of their relative S-matrix cancel out

$$e^{i(\mathrm{p}_1+\mathrm{p}_2)N_f}\prod_{i\neq 1}^{N_a} S^{11}(v_1-v_i)\prod_{i\neq 2}^{N_a} S^{11}(v_2-v_i) = e^{i(\mathrm{p}_1+\mathrm{p}_2)N_f}\prod_{i\neq 1,2}^{N_a} S^{11}(v_1-v_i)S^{11}(v_2-v_i) = 1,$$

and the two particles together effectively scatter with the others by the S matrix

$$S^{21}(v-v_i) = S^{11}(v_1-v_i)S^{11}(v_2-v_i) = \frac{v-v_i-3i}{v-v_i+3i}\frac{v-v_i-i}{v-v_i+i},$$

where $v = (v_1+v_2)/2 = v_1+i$. If the sum of their momenta is real this equation is fine, and the momenta can be part of a solution to the Bethe equations. In terms of rapidities this solution would look like

$$v_1 = v-i, \;\; v_2 = v+i, \;\; v \in \mathbb{R}. \tag{2.3.8}$$

On the other hand, if the sum of our momenta has positive imaginary part we are still in trouble.[8] In this case, since we should avoid coincident rapidities in the Bethe ansatz, the only way to fix things is to have a third particle in the solution, with rapidity

$$v_3 = v_2 + 2i. \tag{2.3.9}$$

As before, if now the total momentum is real the equations are consistent and these three rapidities can form part of a solution. If not, we continue this process and create a bigger configuration, or run off to infinity. These configurations in the complex rapidity plane are known as Bethe strings, illustrated in figure 3. Since our spin chain momentum p has

[6] Here we directly follow the discussion of this topic in [41].

[7] They exist for instance for the Bethe equations with $N_f = 5$, $N_a = 2$.

[8] By rearranging the order of our argument (the particles considered) we do not have to consider the case where the remaining imaginary part is of different sign.

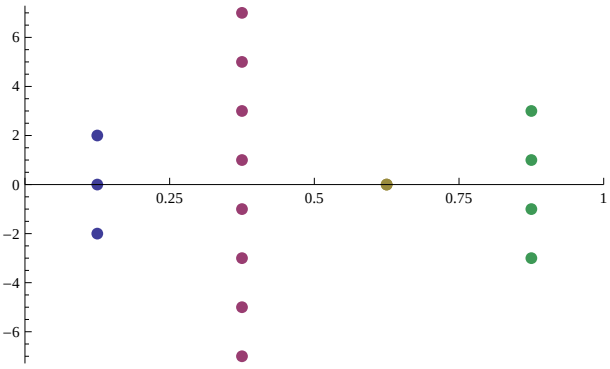


Figure 3: Bethe strings. Bethe strings are patterns of rapidities with spacing $2i$. Here we illustrate strings of length three, eight, one and four, with center 1/8, 3/8, 5/8 and 7/8 respectively.

positive imaginary part in the lower half of the complex rapidity plane and vice versa, strings of any size can be generated in this fashion by starting appropriately far below the real line.[9] Concretely, a Bethe string with Q constituents and rapidity v is given by the configuration

$$\{v_Q\} \equiv \{v - (Q + 1 - 2j)i | j = 1, \ldots, Q\}, \tag{2.3.10}$$

where $v \in \mathbb{R}$ is called the center of the string. Full solutions of the Bethe equation in the limit $N_f \to \infty$ can be built out of these string configurations. Let us emphasize that these string solutions only "exist" for $N_f \to \infty$. At large but finite N_f root configurations are typically only of approprimate string form.

These (Bethe) strings can be interpreted as bound states, having less energy than sets of individual real magnons.[10] For example, the energy of the two-string (2.3.8) is

$$\mathrm{E}_2(v) = \mathrm{E}(v_1) + \mathrm{E}(v_2) = -2J\left(\frac{1}{(v-i)^2+1} + \frac{1}{(v+i)^2+1}\right) = -2J\frac{2}{v^2+2^2}, \tag{2.3.11}$$

which is less than that of any two-particle state with real momenta:

$$\mathrm{E}_2(v) < \mathrm{E}(\tilde{v}_1) + \mathrm{E}(\tilde{v}_2) \quad \text{for } v, \tilde{v}_{1,2} \in \mathbb{R} \quad \text{(real momenta)}. \tag{2.3.12}$$

Similarly, the energy of a Q-string is lower than that of Q separate real particles and is given by

$$\mathrm{E}_Q(v) = \sum_{v_j \in \{v_Q\}} \mathrm{E}(v_j) = -2J\frac{Q}{v^2+Q^2}. \tag{2.3.13}$$

This is most easily shown by noting that

$$\mathrm{E}(v) = J\frac{d\mathrm{p}(v)}{dv}, \tag{2.3.14}$$

[9] In other models the pattern of possible string configurations can be quite complicated, see e.g. chapter 9 of [8] for the XXZ spin chain as a classic example, or [42] and [43] for more involved examples.

[10] The corresponding Bethe wave-function also shows an exponential decay in the separation of string constituents.

and the particularly simple expression for the momentum of a Q-string

$$\mathrm{p}^Q(v) = i \log \frac{v - Qi}{v + Qi}, \tag{2.3.15}$$

as follows by cancelling numerators and denominators in the product $\frac{v_1-i}{\cancel{v_1+i}}\frac{\cancel{v_2-i}}{v_2+i}\dots\frac{v_Q-i}{v_Q+i}$ as indicated.

We have just determined that the possible solutions of the Bethe equations in the limit $N_f \to \infty$ are built out of elementary objects called Bethe strings (a one-string being a normal magnon). Interpreting them as bound states, the spectrum thus obtained is reflected by an appropriate pole in the two-particle S-matrix. This example is not a field theory, but such patterns generically hold there (as well).

So far so good, but ultimately we are interested in thermodynamic limits, meaning we should take $N_f \to \infty$ with $N_a/N_f \leq 1/2$ fixed – the number of magnons goes to infinity as well. In this limit the analysis above is no longer even remotely rigorous since an ever growing product of magnon S-matrices with complex momenta can mimic the role of the pole in our story for example. Still, since such solutions seem rather atypical and at least low magnon density solutions should essentially conform to the string picture, we can hypothesize that 'most' of the possible solutions are made up of string complexes, in the sense that they are the ones that give measurable contributions to the free energy. Indeed in the XXX spin chain there are examples of solutions that do not approach string complexes in the thermodynamic limit [44–46], but nonetheless the free energy is captured correctly by taking only string configurations into account [47]. The assumption that all thermodynamically relevant solutions to the Bethe equations are built up out of such string configurations, and which form these configurations take, goes under the name of the *string hypothesis*. More details and references on the string hypothesis can for example be found in chapter four of [10].

Bethe equations for string configurations

With our string hypothesis for possible solutions in the thermodynamic limit, we would like to group terms in the Bethe equations accordingly – the N_a magnons of a given solution of the Bethe equations should arrange themselves into combinations of string complexes. Denoting the number of bound states of length Q occurring in a given configuration by N_Q we have

$$\prod_{j=1}^{N_a} \to \prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} \prod_{j \in \{v_{Q,l}\}} , \tag{2.3.16}$$

under the constraint

$$\sum_{Q=1}^{\infty} Q N_Q = N_a. \tag{2.3.17}$$

We can then appropriately represent the Bethe equations as

$$e^{i\mathrm{p}_i N_f} \prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} S^{1Q}(v_i - v_{Q,l}) = -1, \tag{2.3.18}$$

where

$$S^{1Q}(v - w_Q) \equiv \prod_{w_j \in \{w_Q\}} S^{11}(v - w_j). \tag{2.3.19}$$

At this point not all N_a Bethe equations are independent anymore, as some magnons are bound in strings – only their centers matter. We already saw that we can get the Bethe equation for the center of a bound state by taking a product over the Bethe equations of its constituents, so that our (complete) set of Bethe equations becomes

$$e^{ip_r^P N_f} \prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} S^{PQ}(v_{P,r} - v_{Q,l}) = (-1)^P, \tag{2.3.20}$$

where

$$S^{PM}(v_P - w) \equiv \prod_{v_i \in \{v_P\}} S^{1M}(v_i - w). \tag{2.3.21}$$

Note that we include the term with $(Q, l) = (P, r)$ in the product above since we took the product in the Bethe equations (2.3.2) to run over all particles. Since $S^{PP}(0) = (-1)^{P^2} = (-1)^P$ however, we could cancel this $(Q, l) = (P, r)$ term against the $(-1)^P$ in the Bethe equations for string configurations if we wanted to.

Physically these expressions represent the scattering amplitudes between the particles indicated by superscripts. These products of constituent S-matrices typically simplify, but their concrete expressions are not important for our considerations (yet); what is important is that they exist and only depend on the centers of the strings, i.e. the overal momenta of the bound state configurations. Combining a set of magnons into a string (bound state) is known as fusion, and the above product denotes the fusion of the corresponding scattering amplitude. You might have encountered similar ideas applied to obtain bound state S-matrices from fundamental ones for instance in [48], here we just did it at the diagonalized level.

2.3.2 Thermodynamics

We now have a grasp on the types of solutions of our Bethe equations in the thermodynamic limit, though this is far from rigorous. We will assume that our classification of possible solutions in terms of strings accurately describes the system in the thermodynamic limit. With this assumption we can proceed as before and derive the thermodynamic Bethe ansatz equations.

We begin with the Bethe equations in logarithmic form, introducing an integer I in each equation which labels the possible solutions

$$-2\pi I_r^P = N_f p^P(v_{P,r}) - i \prod_{Q=1}^{\infty} \prod_{\substack{l=1 \\ l \neq r}}^{N_Q} \log S^{PQ}(v_{P,r} - v_{Q,l}). \tag{2.3.22}$$

We choose to define the integer with a minus sign for reasons we will explain shortly. As by now usual, the solutions to these equations become dense

$$v_i - v_j \sim \mathcal{O}(1/N_f), \tag{2.3.23}$$

and we generalize the integers I to counting functions of the relevant rapidity (momentum). Concretely

$$N_f c^P(u) = -N_f \frac{p^P(u)}{2\pi} - \frac{1}{2\pi i} \sum_{Q=1}^{\infty} \sum_{\substack{l=1 \\ l \neq r}}^{N_Q} \log S^{PQ}(u - v_{Q,l}), \tag{2.3.24}$$

so that

$$N_f c^P(v_l) = I_l^P. \tag{2.3.25}$$

Importantly, in this case we *assume* that the counting functions are monotonically increasing functions of u provided their leading terms are,[11] and here indeed we have

$$\frac{1}{2\pi} \frac{d\mathrm{p}^P(v)}{dv} < 0, \tag{2.3.26}$$

the reason for our sign choice above. Clearly in general we have

$$c(w_i) - c(w_j) = \frac{I_i - I_j}{N_f}. \tag{2.3.27}$$

Introducing particle and hole densities as before, except now in rapidity space, we get

$$\rho^P(v) + \bar{\rho}^P(v) = \frac{dc^P(v)}{dv}, \tag{2.3.28}$$

and explicitly taking the derivative of the counting functions gives us the thermodynamic analogue of the Bethe-Yang equations as

$$\rho^P(v) + \bar{\rho}^P(v) = -\frac{1}{2\pi} \frac{d\mathrm{p}^P(v)}{dv} - K^{PQ} \star \rho^Q(v), \tag{2.3.29}$$

where we implicitly sum over repeated indices, and defined the kernels K as the logarithmic derivatives of the associated scattering amplitudes

$$K^\chi(u) = \pm \frac{1}{2\pi i} \frac{d}{du} \log S^\chi(u), \tag{2.3.30}$$

where χ denotes an arbitrary set of particle labels. The sign is chosen such that the kernels are positive, in this case requiring a minus signs for K^M.[12] As before the Bethe-Yang equations

[11] Here we do not have a convenient positive definite Yang-Yang functional at our disposal. It is not obvious how to prove that these functions are monotonically increasing for given excitation numbers without knowing the precise root distribution, which is what we are actually trying to determine. We may consider it part of the string hypothesis by saying we are not making a mistake in treating the thermodynamic limit as the *ordered* limits $N_f \to \infty$, then $N_a \to \infty$, in which case the statement does clearly hold. A discussion with similar statements can be found on the first page of section six in [49].

[12] Unfortunately we cannot define a notation which uniformizes both the Bethe-Yang equations in the way we did and automatically gives positive kernels.

come in by giving us the hole densities as functions of the particle densities. Varying eqs. (2.3.29) gives

$$\delta\rho^P + \delta\bar{\rho}^P = -K^{PQ} \star \delta\rho_Q, \tag{2.3.31}$$

Writing this schematically as[13]

$$\delta\rho^i + \delta\bar{\rho}^i = K^{ij} \star \delta\rho^j, \tag{2.3.32}$$

after a little algebra we get the variation of the entropy

$$\frac{\delta s}{\delta\rho^j(u)} = \log\frac{\bar{\rho}^j}{\rho^j}(u) + \log\left(1+\frac{\rho^i}{\bar{\rho}^i}\right) \tilde{\star}\, K^{ij}(u), \tag{2.3.33}$$

where again $\tilde{\star}$ denotes 'convolution' from the right (now in u). The variation of the other terms is immediate, and $\delta F = 0$ results in the *thermodynamic Bethe ansatz equations*

$$\log\frac{\bar{\rho}^j}{\rho^j} = \frac{E_j}{T} - \log\left(1+\frac{\rho^i}{\bar{\rho}^i}\right) \star K^{ij}, \tag{2.3.34}$$

where by conventional abuse of notation we dropped the tilde on the 'convolution'. We will henceforth denote the combination $\frac{\bar{\rho}^j}{\rho^j}$ by the *Y functions* Y_j, meaning the TBA equations read

$$\log Y_j = \frac{E_j}{T} - \log\left(1+\frac{1}{Y_i}\right) \star K^{ij}. \tag{2.3.35}$$

Taking into account the generalized form of eqs. (2.3.29) as

$$\rho^i + \bar{\rho}^i = \frac{1}{2\pi}\frac{dp^i}{du} + K^{ij} \star \rho^j, \tag{2.3.36}$$

on a solution of the TBA equations the free energy density is given by

$$f = -T\int_{-\infty}^{\infty} du \frac{1}{2\pi}\frac{dp_j}{du}\log\left(1+\frac{1}{Y_j}\right). \tag{2.3.37}$$

Specifying our schematic notation to eqs. (2.3.29) gives

$$\log Y_P = \frac{\mathrm{E}_P}{T} + \log\left(1+\frac{1}{Y_Q}\right) \star K^{QP}, \tag{2.3.38}$$

and

$$f = T\sum_P \int_{-\infty}^{\infty} du \frac{1}{2\pi}\frac{d\mathrm{p}_P}{du}\log\left(1+\frac{1}{Y_P}\right). \tag{2.3.39}$$

Note the changes of signs due to our conventions on K and p compared to eqs. (2.2.20) and (2.2.21). In stark contrast to the Bose gas, here we are dealing with an infinite set of equations for infinitely many functions, all functions appearing in each equation.

At this point the generalization to an arbitrary model is hopefully almost obvious, with the exception of the string hypothesis which depends on careful analysis of the Bethe(-Yang) equations for a particular model. If we have this however, we can readily determine

[13] Apologies for the immediate mismatch of signs, but this is the general form we would like to take, and cf. eqn. (2.2.11) there is clearly no uniform sign choice.

the complete set of Bethe(-Yang) equations analogous to the procedure to arrive at eqs. (2.3.20). From there we immediately get the analogue of eqs. (2.3.29) by a logarithmic derivative. Note that since we like to think of densities as positive we may have to invert the Bethe(-Yang) equations for a specific particle type to make sure the counting function is defined to be monotonically increasing, just like we did above. This is all we need to specify the general TBA equations (2.3.35) to a given model. Let us quickly do this for our main field theory example of the chiral Gross-Neveu model.

2.4 The chiral Gross-Neveu model

The SU(N) chiral Gross-Neveu model is a model of N interacting Dirac fermions with Lagrangian[14]

$$\mathcal{L}_{cGN} = \bar{\psi}_a i\partial\!\!\!/ \psi^a + \frac{1}{2}g_s^2\left((\bar{\psi}_a\psi^a)^2 - (\bar{\psi}_a\gamma_5\psi^a)^2\right) - \frac{1}{2}g_v^2(\bar{\psi}_a\gamma_\mu\psi^a)^2, \tag{2.4.1}$$

where $a = 1, \ldots, N$ labels the N Dirac spinors. This Lagrangian has $\mathrm{U}(N) \times \mathrm{U}(1)_c$ symmetry, where viewed as an N-component vector the spinors transform in the fundamental representation of U(N), and $\mathrm{U}(1)_c$ denotes the chiral symmetry $\psi \to e^{i\theta\gamma_5}\psi$. The full spectrum of this theory contains $N-1$ SU(N) multiplets of interacting massive fermions, and massless excitations which carry this chiral U(1) charge that decouple completely.[15] We will focus on the SU(2) model.

As a relativistic model the dispersion relation of the fermions is

$$E^2 - p^2 = m^2, \tag{2.4.2}$$

where m is the mass of the fermions. It will be convenient to parametrize energy and momenta in terms of a rapidity u as[16]

$$E = m\cosh\tfrac{\pi u_i}{2}, \;\; p = m\sinh\tfrac{\pi u_i}{2}. \tag{2.4.3}$$

Note that Lorentz boosts act additively on the rapidity, and therefore by Lorentz invariance the two-body S-matrix is a function of the difference of the particles' rapidities only.

The spectrum of the SU(2) chiral Gross-Neveu model contains two species of fermions corresponding to SU(2) spin up and down. This model can be "solved" in the spirit of factorized scattering [50], as discussed for instance in the article by D. Bombardelli [48]. For the SU(2) chiral Gross-Neveu model the upshot is that the scattering of two fermions of equal spin has amplitude

$$S^{ff}(u) = -\frac{\Gamma(1-\frac{u}{4i})\Gamma(\frac{1}{2}+\frac{u}{4i})}{\Gamma(1+\frac{u}{4i})\Gamma(\frac{1}{2}-\frac{u}{4i})}. \tag{2.4.4}$$

[14] Our γ matrices are defined as $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$, $\gamma_5 = \gamma_0\gamma_1$, where $\gamma_{0,1}$ form the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta^{\mu\nu}$ with $\eta = \mathrm{diag}(1,-1)$. Note that γ_5 is Hermitian. As usual $\bar{\psi} = \psi^\dagger\gamma_0$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$.

[15] These facts are far from obvious looking at the Lagrangian, see e.g. section 2.4.1 in [41] for a brief discussion with references. Because of the decoupling of the $U(1)$ mode, g_v is typically put to zero in the chiral Gross-Neveu Lagrangian. Keeping $g_v \neq 0$, however, is useful in demonstrating equivalence to the SU(N) Thirring model.

[16] We choose this unconventional normalization of u to get Bethe-Yang equations in 'the simplest' form. The relation to the rapidity of D. Bombardelli's article [48] is simply $\theta = \pi u/2$.

The relative scattering of fermions with opposite spin is fixed by SU(2) invariance, which leads to a matrix structure matching the R matrix of the XXX spin chain. Diagonalizing the associated transfer matrix results in the Bethe-Yang equations

$$e^{ip_jL}\prod_{m=1}^{N_f} S^{ff}(u_j-u_m)\prod_{i=1}^{N_a} S^{f1}(u_j-v_i) = -1, \tag{2.4.5}$$

$$\prod_{m=1}^{N_f} S^{1f}(v_i-u_m)\prod_{j=1}^{N_a} S^{11}(v_i-v_j) = -1. \tag{2.4.6}$$

which apply in an asymptotically large volume limit, suiting us just fine in the thermodynamic limit. The amplitudes S^{11}, S^{1f} and $S^{f1}(v) = S^{1f}(v)$ are as defined in the previous section in equation (2.3.4). The N_a auxiliary excitations with rapidities v_j correspond to changing the SU(2) spin fermions from up to down; the "vacuum" of the transfer matrix was made up of spin up fermions (cf. spin up states in the XXX spin chain). Note that the equations for the auxiliary excitations become the XXX Bethe equations of the previous section in the limit $u_m \to 0$.

String hypothesis

The two types of fermions of the chiral Gross-Neveu model do not form physical bound states – there is no appropriate pole in the S matrix.[17] However, to take a thermodynamic limit we need to consider finite density states, meaning we will be taking the limit $L \to \infty$, but also $N_f \to \infty$ and $N_a \to \infty$ keeping N_f/L and N_a/N_f fixed and finite. At the auxiliary level we are hence taking the infinite length limit of our XXX spin chain, where we did encounter bound states. The analysis leading to these string solutions is not affected by including the real inhomogeneities u_m corresponding to the physical fermions of the chiral Gross-Neveu model. The only difference is that here the XXX magnons are auxiliary excitations, meaning they carry no physical energy or momentum, and hence the Bethe string solutions lose their interpretation as physical bound states. Nothing changes with regard to them solving the Bethe-Yang equations in the thermodynamic limit however, and we need to take them into account. For the SU(2) chiral Gross-Neveu model we will hence make the string hypothesis that the solutions of its Bethe-Yang equations are given by

- Fermions with real momenta
- Strings of auxiliary magnons of any length with real center

Fusing the Bethe-Yang equations (2.4.5) and (2.4.6) then gives

$$e^{ip_jL}\prod_{m\neq j}^{N_f} S^{ff}(u_j-u_m)\prod_{Q=1}^{\infty}\prod_{l=1}^{N_Q} S^{fQ}(u_j-v_{Q,l}) = -1, \tag{2.4.7}$$

$$\prod_{m=1}^{N_f} S^{Pf}(v_{P,r}-u_m)\prod_{Q=1}^{\infty}\prod_{l=1}^{N_Q} S^{PQ}(v_{P,r}-v_{Q,l}) = (-1)^P, \tag{2.4.8}$$

[17] In our conventions, bound states must have $\mathrm{Im}(u) \in (0, 2i)$, see e.g. [51] or section 2.4.1 of [41].

where

$$S^{\chi Q}(v-w_Q) \equiv \prod_{w_j \in \{w_Q\}} S^{\chi 1}(v-w_j), \qquad \chi = f, 1, \tag{2.4.9}$$

and

$$S^{P\chi}(v_P - w) \equiv \prod_{v_i \in \{v_P\}} S^{1\chi}(v_i - w), \qquad \chi = f, Q. \tag{2.4.10}$$

Thermodynamics

Via the counting functions we get the thermodynamic analogue of the Bethe-Yang equations

$$\rho^f(u) + \bar{\rho}^f(u) = \frac{1}{2\pi}\frac{dp(u)}{du} + K^{ff} \star \rho^f(u) - K^{fQ} \star \rho_Q(u), \tag{2.4.11}$$

$$\rho^P(v) + \bar{\rho}^P(v) = K^{Pf} \star \rho^f(u) - K^{PQ} \star \rho^Q(u), \tag{2.4.12}$$

where again the kernels are defined as in eqn. (2.3.30), positivity of the kernels here requiring minus signs for K^{fP} and K^{Mf}. From our general result above we then find the TBA equations

$$\log Y_f = \frac{E}{T} - \log\left(1 + \frac{1}{Y_f}\right) \star K^{ff} - \log\left(1 + \frac{1}{Y_Q}\right) \star K^{Qf}, \tag{2.4.13}$$

$$\log Y_P = \log\left(1 + \frac{1}{Y_Q}\right) \star K^{QP} + \log\left(1 + \frac{1}{Y_f}\right) \star K^{fP}, \tag{2.4.14}$$

and free energy density

$$f = -T \int_{-\infty}^{\infty} du \frac{1}{2\pi}\frac{dp}{du} \log\left(1 + \frac{1}{Y_f}\right). \tag{2.4.15}$$

The thermodynamics of the chiral Gross-Neveu model (and the XXX spin chain), are determined through an infinite number of integral equations, each directly coupled to all others. Fortunately, this structure can be simplified.

2.5 From TBA to Y system

In problems where there are (auxiliary) bound states the TBA equations can typically be rewritten in a simpler fashion. This is possible for the intuitive reason illustrated in figure 4. Since we obtained all bound state S matrices by fusing over constituents, provided S has no branch cuts the figure shows that

$$\frac{S^{\chi Q+1}(v,u) S^{\chi Q-1}(v,u)}{S^{\chi Q}(v,u+i) S^{\chi Q}(v,u-i)} = 1, \tag{2.5.1}$$

where χ is any particle type and we have reinstated a dependence on two arguments for clarity. We see that (the logs of) our S-matrices satisfy a discrete Laplace equation. Hence the associated kernels would naively satisfy

$$K^{\chi Q}(v,u+i) + K^{\chi Q}(v,u-i) - (K^{\chi Q+1}(v,u) + K^{\chi Q-1}(v,u)) = 0. \tag{2.5.2}$$

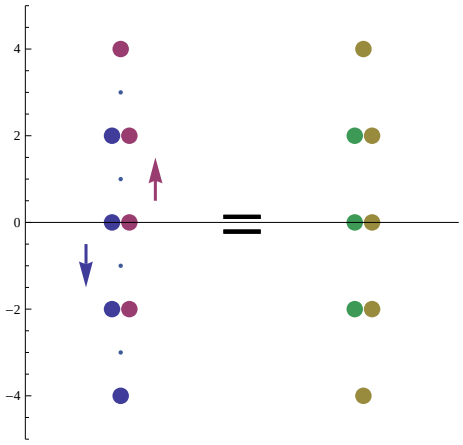


Figure 4: The discrete Laplace equation for strings. Shifting a length Q string configuration up by i and another down by i gives a configuration equivalent to two unshifted strings, one of length $Q+1$ and another of length $Q-1$, here illustrated for $Q=4$. The small dots indicate the position of the rapidities before shifting.

However, when we shift u by $\pm i$ we may generate a pole in $K(v, u+i)$ for some real value of v. This can lead to a discontinuity in integrals involving K such as those in the TBA equations. Therefore we need to understand what exactly we mean by this equation. To do so, let us introduce the kernel s

$$s(u) = \frac{1}{4\cosh\frac{\pi u}{2}}, \tag{2.5.3}$$

and the operator s^{-1} that in hindsight will properly implement our shifts

$$f \star s^{-1}(u) = \lim_{\epsilon\to 0} \left(f(u+i-i\epsilon) + f(u-i+i\epsilon) \right), \tag{2.5.4}$$

which satisfy

$$s \star s^{-1}(u) = \delta(u). \tag{2.5.5}$$

Note that s^{-1} has a large null space, so that $f \star s^{-1} \star s \neq f$ in general; we will see examples of this soon. This kernel can now be used to define

$$(K+1)^{-1}_{PQ} = \delta_{P,Q} - I_{PQ} s, \tag{2.5.6}$$

where the incidence matrix $I_{PQ} = \delta_{P,Q+1} + \delta_{P,Q-1}$, and $\delta_{M,N}$ is the Kronecker delta symbol. This is defined so that

$$(K+1)_{MP} \star (K+1)^{-1}_{PN} = 1_{M,N}, \tag{2.5.7}$$

where 1 denotes the identity in function and index space: $1_{M,N} = \delta(u)\delta_{M,N}$. In other words, the kernel K^{PQ} introduced above is supposed to satisfy

$$K^{PQ} - (K^{PQ+1} + K^{PQ-1}) \star s = s\, I_{PQ}, \tag{2.5.8}$$

which we can prove by Fourier transformation, see appendix A for details. Similarly we have

$$K^{fQ} - (K^{fQ+1} + K^{fQ-1}) \star s = s\, \delta_{Q1}. \tag{2.5.9}$$

Note how the naive picture of eqn. (2.5.2) misses the right hand side of these identities. If a set of TBA equations contains other types of kernels these typically also reduce to something nice after acting with $(K+1)^{-1}$.

Simplified TBA equations

With these identities we can rewrite the auxiliary TBA equations (2.4.14) for the chiral Gross-Neveu model as

$$\log Y_Q = \log(1+Y_{Q+1})(1+Y_{Q-1}) \star s + \delta_{Q,1} \log\left(1+\frac{1}{Y_f}\right) \star s. \tag{2.5.10}$$

This follows from convoluting the equations for $Y_{Q\pm1}$ with s and subtracting them from the equation for Y_Q. Note the remarkable simplification that all infinite sums have disappeared! These TBA equations are not surprisingly known as simplified TBA equations, versus the canonical ones we derived them from.

We should be careful not to oversimplify however. The fact is that $(K+1)^{-1}$ has a null space that is typically of physical relevance. For example, if we take our chiral Gross-Neveu model and turn on a (constant) external magnetic field B coupling to the SU(2) spin of a particle, this would manifest itself as a constant term in the 'energy' of magnons (i.e. a chemical potential), and would lead to a term $\sim B \times P$ in the TBA equation for Y_P, cf. eqs. (2.3.35). Since $c \star s = c/2$ for constant c, such a term is in the null space of $(K+1)^{-1}_{PQ}$ (cf. eqn. (2.5.6)), and hence the simplified TBA equations would not distinguish between different values of this magnetic field. In short, the canonical TBA equations carry more information than the simplified TBA equations. Though this point will come back below in section 3.3, we will not explicitly resolve this technical point here.[18] The extra information required to reconstruct our magnetic field for example, lies in the large u asymptotics of the Y functions, and upon specifying this information our simplified TBA equations are good to go.

The infinite sum in the main TBA equation can also be removed. Noting that similarly to K^{fQ}, K^{Qf} satisfies

$$K^{Qf} - I_{QP} s \star K^{Pf} = s\delta_{Q1}, \tag{2.5.11}$$

we can rewrite the above simplified equations as

$$\log Y_Q - I_{QP} \log Y_P \star s = I_{QP} \log\left(1+\frac{1}{Y_P}\right) \star s + \delta_{Q,1} \log\left(1+\frac{1}{Y_f}\right) \star s. \tag{2.5.12}$$

Integrating with K^{Qf} and using eqn. (2.5.11) we get

$$\log Y_1 \star s = \log\left(1+\frac{1}{Y_Q}\right) \star K^{Qf} - \log\left(1+\frac{1}{Y_1}\right) \star s + \log\left(1+\frac{1}{Y_f}\right) \star s \star K^{1f}, \tag{2.5.13}$$

or in other words

$$\log\left(1+\frac{1}{Y_Q}\right) \star K^{Qf} = \log(1+Y_1) \star s - \log\left(1+\frac{1}{Y_f}\right) \star s \star K^{1f}. \tag{2.5.14}$$

[18] Further discussion can be found in e.g. chapter four of [41].

Figure 5: The TBA structure for the chiral Gross-Neveu model in diagrammatic form. This graph illustrates the coupling between nearest neighbours in the simplified TBA equations (2.5.16) or Y system (2.5.17), where the different colour on the first node signifies the fact that it is 'massive' corresponding to the $\delta_{M,0}$ term in the simplified equations (this is also frequently denoted by putting a $\times$ in the open circle).

The main TBA equation (2.4.13) then becomes

$$\log Y_f = \frac{E}{T} - \log(1+Y_1) \star s, \tag{2.5.15}$$

upon noting that magically enough the Y_f contribution drops out completely thanks to $K^{ff} = s \star K^{1f}$.[19] For uniformity we can define $Y_0 \equiv Y_f^{-1}$ and get

$$\log Y_M = \log(1+Y_{M+1})(1+Y_{M-1}) \star s - \delta_{M,0}\left(\frac{E}{T}\right) \tag{2.5.16}$$

with $Y_M \equiv 0$ for $M < 0$.

Y system

To finish what we started, we can now apply s^{-1} to these equations to get

$$Y_M^+ Y_M^- = (1+Y_{M+1})(1+Y_{M-1}), \tag{2.5.17}$$

where the $\pm$ denote shifts in the argument by $\pm i$; $f^{\pm}(u) \equiv f(u \pm i)$. Note that the energy is in the null space of s^{-1}. These equations are known as the *Y system* [15]. In general, the structure of simplified TBA equations and Y systems can be represented diagrammatically by graphs. For example, in this case eqs. (2.5.16) and (2.5.17) can be represented by figure 5. For more general models the Y system is defined on a certain two dimensional grid, for instance the SU(3) chiral Gross-Neveu model and SU(3) version of the Heisenberg spin chain would have a Y system corresponding to the diagram in figure 6. These diagrams have a group theoretical interpretation. We got extra Y functions for the XXX spin chain and chiral Gross-Neveu model due to the presence of bound states. These bound states of Q particles carry total spin $Q/2$, which we can put into correspondence with the irreducible representations of SU(2). For higher rank symmetry algebras like SU(3), the story is similar: the Y functions correspond to inequivalent non-singlet irreducible representations. The irreducible representations of SU(3) can be represented by Young diagrams of maximal height

[19] To show this we can for example compute the integral in the second term by residues. The cancellation of the complicated scalar factor of the S matrix in the simplified TBA equations appears to be ubiquitous, an observation first made in [15], as an intriguing manifestation of what must be crossing symmetry. Interestingly, at least in some cases we can reverse-engineer the scalar factor from this property [52].

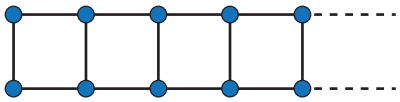


Figure 6: The SU(3) Y system in diagrammatic form.

three. All inequivalent non-singlet ones correspond to diagrams of height two, however, which match the entire diagram of figure 6 if we draw a square around every node.[20]

Let us emphasize again that in this process we lose information at each step along the way: both $(K+1)^{-1}$ and s^{-1} have null-spaces. Therefore the simplified TBA equations are only equivalent to the canonical TBA equations provided we specify additional information on the Y functions such as their large u asymptotics. An alternative but when applicable equivalent specification often encountered in the literature is to give the large Q asymptotics of the Y_Q functions.[21] The Y system requires even further specifications to really correspond to a particular model. For example the Y system for the XXX spin chain is given by dropping Y_0 from the chiral Gross-Neveu Y system altogether, but this is nothing but the chiral Gross-Neveu Y system again, just shifting the label M by one unit.

3 Integrability in finite volume

So far we have used integrability to get an exact description of the large volume limit of our theory, and used this to find a description of its thermodynamic properties in this limit. When the system size is finite however, the notion of an S-matrix – let alone factorized scattering – does not exist, making our integrability approach fundamentally inapplicable. Interestingly however, there is a way around this, allowing us to compute the finite size spectrum of an integrable field theory exactly. Parts of this section directly follow the corresponding discussion in chapter 2 of [41].

3.1 The ground state energy in finite volume

Let us not be too ambitious and begin by attempting to compute the ground state energy of our theory in finite volume. This is possible thanks to a clever idea by Zamolodchikov [12]. To describe this idea let us recall that the ground state energy is the leading low temperature

[20] There are also many integrable models with so-called quantum group symmetry. The representation theory in these cases is more involved, and for instance can result in a maximal spin. Correspondingly, in such cases TBA analysis results in a Y system with finitely many Y functions, see e.g. chapter 7 of [41] for more details. An extensive review on Y systems and so-called T systems can be found in [53].

[21] Already for constant solutions of say the simplified TBA equations of the chiral Gross-Neveu model with $Y_0 \to 0$ there is large ambiguity: for constant Y functions the simplified TBA equations are equivalent to the Y system (of course without rapidities to shift), which is now nothing but a recursion relation fixing everything in terms of Y_1. As will come back below, only one value of this constant corresponds to a solution of the canonical equations with fixed chemical potentials.

contribution to the (Euclidean) partition function

$$Z(\beta, L) = \sum_n e^{-\beta E_n} \sim e^{-\beta E_0}, \qquad \text{as } \beta \equiv \frac{1}{T} \to \infty. \tag{3.1.1}$$

We can compute this partition function with our original quantum field theory by Wick rotating $\tau \to \tilde{\sigma} = i\tau$ and considering a path integral over fields periodic in $\tilde{\sigma}$ with period β. Geometrically we are putting the theory on a torus which in the zero temperature limit degenerates to the cylinder we began with. Analytically continuing $\tilde{\sigma}$ back to τ gives back our original Lorentzian theory. We could, however, also analytically continue $\sigma \to \tilde{\tau} = -i\sigma$. This gives us a Lorentzian theory where the role of space and time have been interchanged with respect to the original model – it gives us its *mirror model*.[22] Putting it geometrically, we could consider Hamiltonian evolution along either of the two cycles of the torus. Note that at the level of the Hamiltonian and the momentum the mirror transformation corresponds to

$$H \to i\tilde{p}, \qquad p \to -i\tilde{H}, \tag{3.1.2}$$

where mirror quantities are denoted with a tilde. To emphasize its role as the mirror volume, let us from now on denote the inverse temperature β by R. In principle we can compute the Euclidean partition function both through our original model at size L and temperature $1/R$ and through the mirror model at size R and temperature $1/L$. These ideas are illustrated in figure 7.

To find the ground state energy of our model then, we could equivalently compute the infinite volume partition function of our mirror model at finite temperature, i.e. its (generalized) free energy $\tilde{F}$ since

$$Z = e^{-L\tilde{F}}. \tag{3.1.3}$$

In fact, cf. eqn. (3.1.1), the ground state energy is related to the free energy density of the mirror model as

$$E_0 = \frac{L}{R}\tilde{F} = L\tilde{f}. \tag{3.1.4}$$

The key point of this trick is that we are working with the mirror model in the infinite volume limit where we can use factorized scattering and the asymptotic Bethe ansatz of the previous section, since any exponential corrections to them can be safely neglected.[23] The

[22] A double Wick rotation leaves a relativistic field theory invariant, and hence we do not really need to carefully make this distinction here. Still, we will occasionally do so for pedagogical purposes. The integrable models encountered in the context of the AdS/CFT correspondence are not Lorentz invariant for instance, meaning the double Wick rotation produces a different model. The term mirror model and mirror transformation were introduced in this context in [19]. Interestingly, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model – the model on which the $\mathrm{AdS}_5/\mathrm{CFT}_4$ TBA is based – can be interpreted as a string itself [54, 55]. The spectrum of this string is thereby related to the thermodynamics of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string, and vice versa [56].

[23] Note again that the mirror of a relativistic model is equal to the original (up to the specific boundary conditions required to compute the same partition function), and therefore the mirror model is immediately integrable as well. In general the conservation laws responsible for factorized scattering are preserved by Wick rotations, so that the mirror theory has many conserved quantities and mirror scattering should factorize. Moreover we can obtain the S-matrix from four point correlations functions via the LSZ reduction formula, and correlation functions can be computed by Wick rotations.

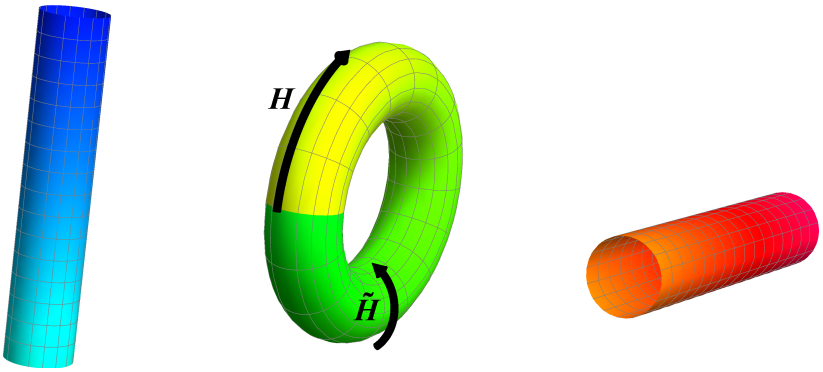


Figure 7: The mirror trick. The partition function for a theory on a finite circle at finite temperature lives on a torus (middle). In the zero temperature limit this torus degenerates and gives the partition function on a circle at zero temperature (left), dominated by the ground state energy. Interchanging space and time we obtain a mirrored view of this degeneration as the partition function of the mirror theory at finite temperature but on a decompactified circle, determined by the infinite volume mirror free energy (or Witten's index).

price we have to pay is dealing with a finite temperature. Fortunately we just learned how to do precisely this, and we can compute our ground state energy from the thermodynamic Bethe ansatz applied to the double Wick rotated (mirror) model.

We should be a little careful about the boundary conditions in our model however. Where fermions are concerned the Euclidean partition function is only the proper statistical mechanical partition function used above, provided the fermions are anti-periodic in imaginary time. Turning things around, if the fermions are periodic on the circle then from the mirror point of view they will be periodic in imaginary time, so that our goal in the mirror theory is not to compute the standard statistical mechanical partition function but rather what is known as Witten's index

$$Z_W = \mathrm{Tr}\left((-1)^F e^{-L\tilde{H}}\right), \tag{3.1.5}$$

where F is the fermion number operator. This means we are adding $i\pi F/L$ to the Hamiltonian – a constant imaginary chemical potential for fermions.[24]

We should also note that the mirror transformation actually has a nice meaning on the rapidity plane, provided we adapt it slightly. From our discussion above, we see that the energy and momentum of a particle should transform as

$$E \to i\tilde{p}, \qquad p \to -i\tilde{E}, \tag{3.1.6}$$

which leaves its relativistic dispersion relation $E^2 - p^2 = m^2$ invariant. This means we can parametrize $\tilde{E}$ and $\tilde{p}$ exactly as before ($E(u) = \cosh\frac{\pi u}{2}$ and $p(u) = \sinh\frac{\pi u}{2}$), but let us say now in terms of a mirror rapidity $\tilde{u}$. We can then wonder what the relation between u and

[24]Continuing along these lines, if we were to consider quasi-periodic boundary conditions instead of (anti-)periodic boundary conditions a more general operator enters in the trace, which leads to more general chemical potentials. For details see e.g. chapter two and four of [41].

$\tilde{u}$ should be. By definition we want

$$E(u) = ip(\tilde{u}), \qquad p(u) = -iE(\tilde{u}). \tag{3.1.7}$$

Now we recall that sines and cosines are related by shifts of $\pi/2$, which in the hyperbolic case tells us that

$$E(u-i) = i\sinh\tfrac{\pi u}{2} = -ip(-u), \qquad p(u-i) = -i\cosh\tfrac{\pi u}{2} = -iE(\pm u). \tag{3.1.8}$$

Hence we see that if we identify $-\tilde{u} = u - i$, we get what we want. In the literature you will however typically encounter the transformation $u \to u + i$ ($\theta \to \theta + i\frac{\pi}{2}$ in the standard relativistic rapidity parametrization) which is quite convenient and we will use from here on out.[25] Here the rapidity on the right hand side actually implicitly refers to the mirror rapidity $\tilde{u}$, matching our story so that

$$u \to \tilde{u} + i, \text{ i.e. } \tilde{u} = u - i. \tag{3.1.9}$$

This means that in addition to what we are doing here, people frequently do a parity transformation in between. For parity invariant theories this does absolutely nothing, and even if a theory is not parity invariant, we could simply proceed this way and compute things in the parity flipped theory, reverting back only at the final stage.

Applying the above discussion to the chiral Gross-Neveu model, we see that we can compute its ground state energy on a circle of circumference L by taking our derivation of the free energy above, replacing the length L by the mirror length R, replacing the temperature T by the inverse length $1/L$, and adding a constant term $i\pi/L$ to the dispersion relation for the fermions. The ground state energy is then given by

$$E_0 = -\int_{-\infty}^{\infty} du \frac{1}{2\pi}\frac{dp}{du}\log\left(1 + Y_0\right), \tag{3.1.10}$$

where Y_0 satisfies the (mirror) TBA equations

$$\log Y_M = \log(1 + Y_{M+1})(1 + Y_{M-1}) \star s - \delta_{M,0}(LE + i\pi), \tag{3.1.11}$$

together with the $Y_{M>0}$. Note the added $i\pi$ in line with the periodicity of the fermions in imaginary mirror time.

3.2 Tricks with analytic continuation

At this point we have actually done something quite impressive: we have found a system of equations we can solve (admittedly numerically) to find the exact finite volume ground state energy of a two dimensional field theory. It would be great if we could extend this approach to the entire spectrum. If we look back at our arguments however, we are immediately faced with a big conceptual problem; the mirror trick and infinite volume limit work nicely

[25] While widely used, the name mirror transformation is appropriate for the case we started with, as you can readily convince yourself of by drawing a picture in the complex (σ, τ) plane. What does the second transformation do?

precisely for the ground state and the ground state only! Still it is hard to believe that a set of complicated TBA equations knows about the ground state only, especially since they are derived from the mirror Bethe-Yang equations which are just an analytic continuation away from describing the *complete* large volume spectrum. In this section we will take an approach often taken in physics; we will (try to) analytically continue from one part of a problem to another, in this case from the ground state energy to excited state energies. The idea that excited states can be obtained by analytic continuation is an old one, discussed in e.g. [57] in the case of the quantum anharmonic oscillator.

3.2.1 A simple example

Before moving on, we would like to motivate these ideas and illustrate them on a simple quantum mechanical problem[26]

$$H\psi = E\psi, \quad \text{with} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2.1}$$

After considerable effort we realize that the spectrum in this model is given by

$$E(\lambda) = \pm\sqrt{1+\lambda^2}, \tag{3.2.2}$$

and hence the ground state energy is $-\sqrt{1+\lambda^2}$. Allowing ourselves to analytically continue in the coupling constant we realize that the equation for the ground state energy has branch points at $\lambda = \pm i$. As a consequence, analytically continuing around either of these branch points and coming back to the real line we do not quite get back the ground state energy, but rather the energy of the excited state. This is illustrated in figure 8(a). The message we can take away from this [58] is that by analytically continuing a parameter around a "closed contour" – meaning we come back to the "same" value though not necessarily on the same sheet – we end up back at the same problem although our eigenvalue may have changed. As we are still dealing with the same problem, if the eigenvalue has changed under analytic continuation it must have become one of the other eigenvalues. Note that this does not imply all eigenvalues can be found this way – the spectrum may split into distinct sectors closed under analytic continuation.

Let us now forget about the description of this problem in terms of linear algebra, and suppose for the sake of the argument that in solving our spectral problem we had obtained

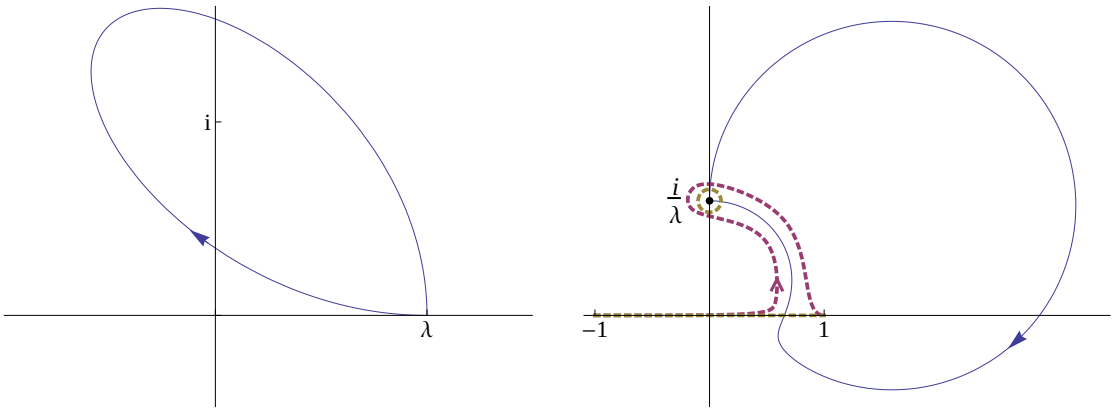
$$E(\lambda) = -\int_{-1}^{1} dz\, \frac{1}{2\pi i} f(z) g(z) - 1, \tag{3.2.3}$$

where

$$f(z) = \frac{1}{z - i/\lambda}, \quad \text{and} \quad g(z) = 2\lambda\sqrt{1-z^2}. \tag{3.2.4}$$

We can determine that this integral has branch points at $\lambda = \pm i$ without knowing anything about $f(z)$ other than that it is meromorphic with a single pole at i/λ. Conceptually we consider $g(z)$ to be some nice known function, while f is not explicitly known. Analytically

[26] This nice example can be found in slides of a talk by P. Dorey at IGST08 [58].



(a) (b)

Figure 8: Analytic continuation. The left figure shows the analytic continuation of λ (blue) around the branch point at i, corresponding to flipping the sign of $-\sqrt{1+\lambda^2}$ upon returning to the real line. The right figure shows the corresponding movement of the pole at i/λ (blue) which drags the integration contour (red, dashed) in eqn. (3.2.3) along with itself for continuity. Upon taking the integration contour back to the real line we retain a residual contribution (yellow, dashed).

continuing the integral in λ we get a function that is well defined everywhere except for the half-lines $i\lambda > 1$ and $i\lambda < -1$ where the pole moves into the integration domain. Continuing around the point $\lambda = i$ as in figure 8(a), nothing happens when we first cross the line $\mathrm{Re}(\lambda) = 0$ but when we cross the second time, the pole moves through the integration contour on the real line and drags the contour along, as illustrated in figure 8(b). We can rewrite the resulting contour integral in terms of the original one by picking up the residue, giving

$$E^c(\lambda) = -\int_{-1}^{1} dz\, \frac{1}{2\pi i} f(z)g(z) + g(i/\lambda) - 1. \tag{3.2.5}$$

Since $E^c(\lambda) - E(\lambda) = g(i/\lambda) \neq 0$ there must be a branch point inside the contour. In this integral picture we do not need to know the precise analytic expression of E or f to determine the expression for the excited state energy. All we need to know is the pole structure of f relative to the integration contour.

3.2.2 Analytic continuation of TBA equations

Inspired by this example, we can try to analytically continue our expression for the ground state energy, eqn. (3.1.10), in some appropriate variable and see whether we encounter any changes in the description. We could try continuing in the mass variable of the chiral Gross-Neveu model for example. This approach to excited states in the TBA was proposed and successfully applied to what is known as the scaling Lee-Yang model in [13]. The authors there observed that in the process of analytic continuation the Y functions solving the TBA equations undergo nontrivial monodromies. They moreover noted that changes in the form of the TBA equations are possible if singular points of $1 + 1/Y$ move in the complex plane

during the analytic continuation. These changes are analogous to the changes in the energy formula of our example above. Here the integral is a typical term on the right hand side of the TBA equations

$$y(u) \equiv \log\left(1+\frac{1}{Y}\right) \star K(u), \tag{3.2.6}$$

where we recall that $\star$ denotes (right) convolution on the real line. If there is a singular point

$$Y(u^*) = -1, \tag{3.2.7}$$

and its location u^* crosses the real line during the analytic continuation, we can pick up the residue just as in our simple example to get

$$y^c(u) = \log\left(1+\frac{1}{Y}\right) \star K(u) \pm \log S(u^*, u), \tag{3.2.8}$$

where we recall that $K(v,u) = \frac{1}{2\pi i}\frac{d}{dv}\log S(v,u)$ and the sign is positive for singular points that cross the contour from below and negative for those that cross it from above. If Y vanishes at a particular point, this leads to the same considerations, just resulting in an opposite sign.[27] If we wanted to do this at the level of the simplified equations, all we need is the S-matrix associated to s:

$$S(u) = -\tanh\frac{\pi}{4}(u-i). \tag{3.2.9}$$

The energy itself is also determined by an integral equation in the TBA approach, meaning it can change explicitly as well as implicitly through the solution of a changed set of TBA equations.

The upshot of this is that we obtain excited state TBA equations that differ from those of the ground state by the addition of $\log S$ terms, which we will call *driving terms*. It should not matter whether we consider this procedure at the level of the canonical equations or at the level of the simplified equations, and indeed the results agree because of the S-matrix analogue of identities like eqn. (2.5.8).

3.3 Excited states and the Y system

The case of the Y system is a bit more peculiar, since the distinguishing features of an excited state completely disappear. This is because the S-matrix (3.2.9) vanishes under application of s^{-1}. From this we see that whatever excited state TBA equations we obtain by the above reasoning, the Y system equations are the same as those of the ground state: the Y system is *universal*.[28] The important distinction is that as we just said the Y functions for excited states have singular points. If there are no further singularities like branch cuts (which we would expect to be universal features of a model rather than state dependent), specifying the

[27] The singular points of different Y functions in the complex plane are typically related. A driving term arises from a special point u^* for a Y function on the right hand side of a TBA equation. Since this term typically has poles at $u^* \pm i$, however, this shifted point corresponds to a zero or pole for the Y function on the left hand side. Analyzing the Y system (discussed just below) we arrive at the same conclusion.

[28] Exemptions to this rule can arise under very specific circumstances, see e.g. [59] and chapter seven of [41].

number of simple poles and zeroes of all Y functions in the strip between i and $-i$ is almost enough to 'integrate' the Y system back to the simplified TBA equations. First, however, we need to address the fact that different physical models can have the same Y system. This also brings us back to the discussion of information loss in the simplifying steps of section 2.5.

Asymptotics of Y functions

For concreteness, let us consider the simplified TBA equations (2.5.16) for the chiral Gross-Neveu model. The distinguishing feature of these equations with respect to say the XXX ones is the energy contribution to Y_0. This term leads to $\log Y_0 \sim -e^{\pi/2|u|}/T$ at large $|u|$, meaning Y_0 goes to zero quite rapidly. If we take these asymptotics as given and assume Y_0 is analytic in the strip between i and $-i$, for the time being interpreting Y_1 as some given external function, we can 'integrate' the Y system equation $Y_0^+ Y_0^- = (1 + Y_1)$ to the associated simplified TBA equation. Namely

$$Y_0 = e^{-E/T} e^{\log(1+Y_1)\star s} \tag{3.3.1}$$

satisfies the Y system (note again that $e^{-E/T}$ drops out of this), has the right asymptotics, and is analytic, which by Liouville's theorem means it is unique (the difference with any other function with these properties is zero).

To get the simplified TBA equations for the remaining Y functions, which have no energy terms, it turns out we should demand constant asymptotics $Y_N \to \hat{Y}_N$. These constants are all recursively determined by one of them, e.g. $\hat{Y}_1$, by the constant limit of the Y system (where $\hat{Y}_0 = 0$ in line with its asymptotics), i.e.

$$\begin{aligned} \hat{Y}_2 &= \hat{Y}_1^2 - 1, \\ \hat{Y}_{N+1} &= \frac{\hat{Y}_N^2}{1 + \hat{Y}_{N-1}} - 1, \quad N > 1. \end{aligned} \tag{3.3.2}$$

A simple solution to this set of equations is $Y_M = M(M+2)$, essentially due to the identity $M^2 = (M+1)(M-1) + 1$. We can generalize this solution to

$$\hat{Y}_M = [M]_q [M+2]_q, \tag{3.3.3}$$

where we introduced the so-called q numbers

$$[M]_q = \frac{q^M - q^{-M}}{q - q^{-1}}, \tag{3.3.4}$$

which retain the property $[M]_q^2 = [M+1]_q [M-1]_q + 1$ for any $q \in \mathbb{C}$. In the limit $q \to 1$, $[N]_q \to N$ again. Since everything is recursively fixed by $\hat{Y}_1 = [3]_q$ and by picking q appropriately we can make $[3]_q$ any complex constant, this is the general constant solution of our Y system. Given a value of $\hat{Y}_1$ and hence all $\hat{Y}_M$, the expression for the associated full Y functions as the right hand sides of their TBA equations follows uniquely from analyticity and the Y system as it did for Y_0.

To fix the constant asymptotic of Y_1 we can feed our constant "solution" in to the canonical TBA equations, where now integration with the kernels amounts to multiplication by their normalizations. Then performing the infinite sums in the canonical TBA equations we get a set of equations that admits only one value for $\hat{Y}_1$. In our chiral Gross-Neveu case this fixes the asymptotes of $Y_{M>0}$ to be $M(M+2)$. If we had included a nontrivial chemical potential μ for the spin down fermions in a thermodynamic picture, or cf. footnote 24 double Wick rotated quasi-periodic boundary conditions, we would instead be required to take a different constant q number solution with $\log q \sim \mu$, showing the physical interpretation of these constant asymptotics.[29] This link between chemical potentials and asymptotics actually allows us to move between canonical TBA equations and simplified equations plus (constant) asymptotics.

As mentioned earlier, the XXX spin chain has the same Y system, but different (simplified) TBA equations. These simplified TBA equations would follow along the same lines, but with different asymptotics. Similarly, the $i\pi$ contribution in eqs. (3.1.11) affects the asymptotics relative to eqs. (2.5.16).

Poles and zeroes of Y functions

Now that we have seen how to get basic simplified TBA equations from a Y system, let us try to add driving terms. To do so, we need to know the simple poles and zeroes of the Y functions. Provided we are given this data, we can explicitly factor out poles and zeroes of Y via products of $t(u) = \tanh \frac{\pi}{4}u$ and $1/t$. In other words for a Y function with poles at ξ_i and zeroes at χ_j we define

$$\check{Y}(u) = \frac{\prod_j t(u-\chi_j)}{\prod_i t(u-\xi_i)} Y(u), \tag{3.3.5}$$

which is analytic. We now start from the schematic Y system $Y^+Y^- = \mathcal{R}$, which implies also $\check{Y}^+\check{Y}^- = \mathcal{R}$ because $t^+t^- = 1$. Morever, since $\check{Y}$ is analytic and has the same asymptote as Y because $t(u)$ asymptotes to one, we are essentially in the situation we had above (the relation of $t(u)$ to $S(u)$ of eqn. (3.2.9) is not accidental). By our previous analysis we get

$$\check{Y} = e^{\log \mathcal{R} \star s}, \tag{3.3.6}$$

so that

$$Y = \frac{\prod_i t(u-\xi_i)}{\prod_j t(u-\chi_j)} e^{\log \mathcal{R} \star s}, \tag{3.3.7}$$

where we should include $e^{-E/T}$ as before if necessary. This is precisely of the form of a simplified excited state TBA equation. To reiterate, this formula by definition gives the Y system upon applying s^{-1}, and has the right poles, zeroes, and asymptotics, making it our unique desired answer. For more complicated TBA equations with branch cuts we would need to know the discontinuities of the Y functions across the cuts, in addition to poles, zeroes and asymptotics, but morally we would do the same thing.

[29] More details can be found in e.g. chapters 2 and 4, and appendix A.4 of [41]. In particular, evaluating the infinite sums actually requires an $i\epsilon$ prescription in case of nonzero chemical potential.

In short, by supplying analyticity data in the form of poles, zeroes, and asymptotics, we can derive a set of integral equations of simplified TBA form, with precisely the expected type of energy and driving terms. Some form of integral equations is of course useful, as they can typically be iteratively solved, perhaps by starting from a seed solution in some part of parameter space (an asymptotic solution), which should in particular include appropriate starting values for the zeroes and poles. The notion that analytic properties might "label" excited states also appears in e.g. the discussion of the "Bethe ansatz" for the harmonic oscillator in the article by F. Levkovich-Maslyuk [40].

The Y system and its universality are closely related to other approaches of obtaining equations that describe excited state energies. In some cases it is possible to construct a functional analogue of the Y system directly, as discussed in the article by S. Negro [16]. If we can then get satisfactory insight into the analytic structure of the corresponding objects, we can 'integrate' these functional relations in the above spirit to obtain integral equations describing the energy of excited states [60, 61, 14, 62]. Depending on how these functional equations are 'integrated' we can obtain equations of TBA form but also various other forms that can be more computationally efficient. The latter equations generically go under the name of "non-linear integral equations" [60], but depending on the context are also called "Klümper-Pearce" [63, 64, 60, 65] or "Destri-de Vega" [66, 67] equations. While not obvious from their form, when different types of equations are possible they should of course be equivalent [68, 69].

3.4 Lüscher formulae

In general, we may wish to use an amalgamation of the above ideas to find excited state TBA equations, in the form of something which we will refer to as the contour deformation trick. The basic idea goes as follows. We will find a candidate solution of the Y system for an excited state with some limited regime of applicability. We then assume that the form of the TBA equations for an excited state is uniform and does not change outside of the regime of applicability of our candidate solution. Next, drawing lessons from the analytic continuation story above we expect that the only changes in the equation should be the addition of possible driving terms. Furthermore, although our limited solution only gives us a static picture, we expect that we can qualitatively view these terms as if coming from singular points that crossed the integration contour. Since in this picture such singular points would have dragged the contour along with them, we expect that an excited state TBA equation should be of the same form as the ground state, except with modified integration contours. Analyzing the analytic structure of the candidate solution will allow us to consistently define these contours in such a way that the TBA equations are satisfied, and by taking the integration contours back to the real line we can explicitly pick up the corresponding driving terms. Coming back to our simple example, it would be as if internal consistency of the problem (perhaps in the form of some other equation) told us that the natural integration contour for the excited state was not the interval $(-1, 1)$, but a contour that starts at one and finishes at minus one while enclosing i/λ between itself and the real line. Such a contour is of course equivalent

to the red contour in figure 8(b) obtained by direct analytic continuation.

Through a bit of physical reasoning we can obtain a candidate solution of the TBA equations that should describe an excited state. If we take our theory at face value as a field theory on a cylinder, it is natural to expect the energy of states to get corrections from virtual particles travelling around the circle, a phenomenon investigated in particular by Lüscher [70]. Concretely, Lüscher showed how polarization effects lead to mass corrections for a standing particle in massive quantum field theory in a periodic box, computing their effect to leading order in e^{-mL} where m is the mass of the particle and L the size of the periodic box [70]. These (leading order) corrections come in two types illustrated in figure 9. The first of these is the so-called μ term corresponding to the particle decaying into a pair of virtual particle which move around the circle (in two dimensions) and recombine, while the second is the F-term which corresponds to a virtual particle loop around the circle which involves scattering with the physical particle.

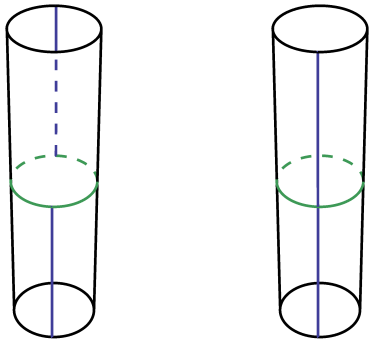


Figure 9: The Lüscher μ- and F-term. On the left we have the decay of a physical particle (blue) into a pair of virtual particles (green) which fuse to a physical particle on the other side of the cylinder, while the right shows the scattering of a virtual particle with the physical particle as it loops around the cylinder.

Generalizing these ideas to moving particles and interacting multi-particle states based on the original diagrammatic methods of [70] seems daunting. In the context of simple relativistic integrable models, however, Lüscher's formulae readily follow by explicitly expanding the TBA equations in the large volume limit. By carefully generalizing the expansions in such models to interacting multi-particle states we can try to obtain a type of generalized Lüscher's formulae. To leading order this energy correction takes the form [71]

$$\Delta E = -\sum_{Q} \int_{-\infty}^{\infty} \frac{d\tilde{p}}{2\pi} e^{-\widetilde{\mathcal{E}} L} \lambda_{Q,1}(\tilde{p}|\{p_j\}), \tag{3.4.1}$$

where we have indicated double Wick rotated (mirror) quantities by a tilde to show their origin, though the distinction will not matter for us here. This is the multi-particle generalization of the contribution corresponding to the F term on the right side of figure 9. In many integrable models the μ term does not appear to show up at leading order for most states. In this formula, $\lambda_{Q,1}(\tilde{p}|\{p_j\})$ denotes the eigenvalue of the transfer matrix for the state of

the integrable model under consideration, with its auxiliary space taken to be the mirror (double Wick rotated) representation for a mirror particle of type Q. In other words, the energy shift is given by scatter any possible virtual particle[30] with the physical excitations of our large volume state, and summing over all of them, weighed by $e^{-\widetilde{\mathcal{E}}L}$.

Now we argued above that the excited state TBA equations should differ from the ground state ones by a set of driving terms, but should otherwise be of the exact same form. Considering the energy formula (3.1.10) in this light, we realize that at large mass or large volume the Y_f function should be small due to the $-L\widetilde{\mathcal{E}} = -mL\cosh\pi u/2$ term in their canonical TBA equations. Expanding the energy formula for small Y_0 and comparing this to the leading weak coupling correction (3.4.1), where for the chiral Gross-Neveu model there is no sum over Q since there are no physical bound states, for an excited state described by a set of rapidities $\{u_i\}$ it is natural to identify

$$Y_0^o(\tilde{u}) = e^{-E(\tilde{u})L}\prod_{i=1}^{N_f} S^{ff}(\tilde{u},u_i)\lambda(\tilde{u}|\{u_i\}), \tag{3.4.2}$$

where the tilde is a label to indicate that the associated quantities are to be evaluated in the mirror theory, and λ refers to the XXX spin chain transfer matrix eigenvalue without the scalar factor S^{ff}, which we hence have to reinstate to describe our chiral Gross-Neveu model. The superscript o indicates that this is an asymptotic solution that only applies to leading order in e^{-EL}. One immediate promising feature of this formula is that if we analytically continue this function from the mirror theory to the physical theory we are interested in, this precisely looks like the right hand side of the Bethe equations, and we get that asymptotically

$$Y_0^o(u_i^*) = -1, \tag{3.4.3}$$

the $*$ denoting that we have analytically continued. This precisely corresponds the kind of singular point we encountered in our general discussion around eqn. (3.2.7)! In fact, assigning appropriate driving terms to these singular points precisely results in an energy formula of the form

$$E = \sum_{i=1}^{N_f} E(p_i) - \int du\,\frac{1}{2\pi}\frac{dp}{du}\log\left(1+Y_0\right)\,, \tag{3.4.4}$$

where $E(p_i)$ is the asymptotic energy of the ith particle (recall that p evaluated on an analytically continued rapidity is just $-iE$). Of course there can be further modifications to this energy formula depending on possible further singular points of Y_0, see for instance [72] for a situation with rather involved analytic properties. Actually, the auxiliary equations could change as well, leading us to wonder what asymptotic solution we should consider there. Not going into technical details, we hope the following sounds reasonable. The auxiliary Y_Q functions are physically associated to the Bethe-string solutions of the XXX spin chain (with inhomogeneities), which as we discussed are bound states, and their S-matrices can be found by fusion. We could construct transfer matrices based on each of the bound state S-matrices labeled by the string length P, and find their eigenvalues. By construction these objects will

[30] At least pictorially it is clear that a virtual particle is like a regularly propagating mirror particle.

satisfy a relation similar to, but slightly more complicated than the one for the diagonalized scattering amplitudes of eqn. (2.5.1), and by using these relations one can consistently express the $Y_{P>0}$ in terms of these bound state transfer matrix eigenvalues. This then gives us a full asymptotic solution, and we can analyze its analytic properties to find excited state TBA equations whose solution we can extend beyond the asymptotic regime.

3.5 $Y/T/Q$-system and nonlinear integral equations

The structure of fusion relations between bound state transfer matrix eigenvalues actually relates nicely to a structure that is known as the T system, a system encountered in S. Negro's article [16] in a particular model. Let us go over the basic story, avoiding formulas. The T system is a set of equations known as Hirota equations for a set of T functions, functions of the rapidity (momentum) defined on a grid with a border one wider than the Y system on all sides. The identification between the Ys and the Ts admits gauge transformations on the Ts, but in an appropriate gauge the asymptotic Y functions are expressed in terms of asymptotic T functions, for which the (asymptotic) T system becomes precisely equivalent to the fusion relations of the transfer matrix eigenvalues. The T system is a generic rewriting of the Y system however, which applies beyond the asymptotic limit. Its gauge freedom actually proves useful, as one can (try to) shift the analytic properties of the Y functions that we require from the TBA, between the various T functions. Doing so appropriately, we can represent the (typically infinite set of) T functions in terms of a set of much simpler elementary functions known as Q functions with transparent analytic properties. Turning the resulting algebraic equations plus analyticity constraints back into integral equations for these "fundamental" variables gives a set of nonlinear integral equations for a finite number of functions, of the general Klümper-Pearce–Destri-de Vega type mentioned above. This hence provides a means of rewriting the TBA equations in a simpler form in these more complicated cases with infinitely many Y functions. In the context of integrability in AdS/CFT these equations are known as the quantum spectral curve [29]. S. Negro's article [16] discusses that deriving such Y, T, or Q systems and their analytic properties from first principles is possible in particular models. While a highly involved problem, doing so in a particular model would provide a great check on the chain of reasoning involved in the TBA approach (for excited states in models with bound states). For the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string first steps in this direction were made in [73].

4 Conclusion

The thermodynamic Bethe ansatz is an important technical tool with applications ranging from (but not limited to) describing the thermodynamic properties of one dimensional spin chains to computing the spectra of integrable field theories on a cylinder. In this article we provided an introduction to the basic ideas behind this method, and applied them in a number of illustrative and representative examples. We started from the simplest Bethe ansatz integrable model – free electrons – where we introduced the thermodynamic limit

and the concept of density of states and holes and their relation via momentum quantization conditions. The stationarity of the free energy in thermodynamic equilibrium resulted in a simple algebraic equation, whose solution gave the famous Fermi-Dirac distribution. We then applied the same ideas with the free particle momentum quantization condition replaced by more complicated Bethe(-Yang) equations, to describe the thermodynamics of the Bose gas, XXX spin chain, and chiral Gross-Neveu model. These latter two models required us to introduce a string hypothesis describing the possible solutions of the Bethe equations in the thermodynamic limit. The stationarity condition now results in one or or an infinite number of coupled integral equations – the TBA equations – for the Bose gas, and XXX spin chain and chiral Gross-Neveu model respectively. We discussed how such infinite sets of TBA equations can be simplified and ultimately reduced to a Y system together with analyticity data, including technical details on integration kernel relations presented in an appendix. We then moved on to using the same ideas to describe the ground state energy of integrable field theories in finite volume via the mirror trick of interchanging space and time, and how these ideas can be adapted and applied to excited states. The Y system structure is the same for all such excited states, and we discussed the analyticity data required to link a Y system to a given model and within that to a given state. We also briefly discussed the basics of and some tips on numerically solving TBA equations. The conceptual background we discussed and applied to our concrete examples make up the essence of the TBA approach, and as such can be applied to (m)any other integrable model(s).

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A Integral identities

In eqn. (2.5.8) of section 2.5, we claimed that the kernels K^{MN} satisfy

$$K^{PQ} - (K^{PQ+1} + K^{PQ-1}) \star s = s\, I_{PQ}. \tag{A.1}$$

We also made claims regarding $K^{fQ} = K^{Qf}$, namely

$$K^{fQ} - (K^{fQ+1} + K^{fQ-1}) \star s = s\, \delta_{Q1}. \tag{A.2}$$

We can prove these by Fourier transform. We begin by noting that similarly to the fused XXX momentum of eqn. (2.3.15),

$$S^{fQ}(u) = \frac{u+iQ}{u-iQ}, \quad K^{fQ}(u) \equiv -\frac{1}{2\pi i}\frac{d}{du}\log S^{1Q}(u) = \frac{1}{\pi}\frac{Q}{Q^2+u^2}. \tag{A.3}$$

Note that $S^{11} = 1/S^{f2}$, but that we defined the kernels with opposite sign, so $K^{11} = K^{f2}$. Now the Fourier transform of K^{fQ} ($Q \geq 1$) is

$$\hat{K}^{fQ}(k) \equiv \int_{-\infty}^{\infty} du e^{iku} K^{fQ}(u) = e^{-|k|Q}, \tag{A.4}$$

while

$$\hat{s}(k) = \frac{1}{2\cosh k}. \tag{A.5}$$

In Fourier space, identity (A.2) is now simply an equality between functions. The identity for K^{QM} similarly follows by its definition as a sum over string states ($K^{f0} = 0$)

$$K^{QM}(u) \equiv \sum_{\text{strings}} K^{11} = \sum_{\text{strings}} K^{f2} \tag{A.6}$$

$$= K^{f(Q+M)}(u) + K^{f(M-Q)}(u) + 2\sum_{i=1}^{Q-1} K_{M-Q+2i}(u) \tag{A.7}$$

$$= K^{f(Q+M)}(u) + K^{f(|M-Q|)}(u) + 2\sum_{i=1}^{\min(M,Q)-1} K_{|M-Q|+2i}(u) \tag{A.8}$$

which we get by combining appropriately shifted numerators and denominators in the product of S matrices underlying these kernels. Its Fourier transform, cf. eqn. (A.4), is

$$\hat{K}^{QM} = \sum \hat{K}^{fX} = \coth|k| \left(e^{-|Q-M||k|} - e^{-(Q+M)|k|}\right) - \delta_{Q,M}, \tag{A.9}$$

from which eqn. (A.1) follows.

B Numerically solving TBA equations

We mentioned in 2.2 that we can numerically solve TBA equations by iterations. Let us consider the general form of a TBA equation

$$\log Y_j = \log\left(1 + \frac{1}{Y_k}\right) \star K_{kj} + a_j, \tag{B.1}$$

where the a denote a set of driving terms, including for instance the energy term in eqn. (2.2.20). To solve these equations by iterations, we start with some guess for the Y function(s) as a seed – the $Y_j^{(0)}$ – and use these initial functions to compute the right hand side of the TBA equations. We then use this to define the updated $Y_j^{(1)}$, or more generally

$$\log Y_j^{(n+1)} = \log\left(1 + \frac{1}{Y_k^{(n)}}\right) \star K_{kj} + a_j. \tag{B.2}$$

In practice we hope these iterations converge to a stable solution.[31] Of course, the trick lies in the technical implementation of this basic concept, which is a bit of an art.

First, a good guess for the initial Y functions will at the very least speed up the process. If we wanted to solve the Bose gas equations (2.2.20), for instance, in a low temperature regime a good guess would be $\epsilon^{(0)}(p) = E(p)$. Second, depending on the details of the equations and kernels, nothing guarantees that eqn. (B.2) will converge fastest. For instance, it may be advantageous to consider

$$\log Y_j^{(n+1)} = x \left(\log \left(1 + \frac{1}{Y_k^{(n)}} \right) \star K_{kj} + a_j \right) + (1-x) \log Y_j^{(n)}, \tag{B.3}$$

for some $0 < x \leq 1$, cf. e.g. section 2 of [13]. This is mostly useful if we need to run similar equations many times, since finding a suitable value for x through experimentation takes time as well. Third, the convolution computations can typically be sped up by means of (fast) Fourier transform (FT), i.e. we compute the convolution $f \star g$ as $\mathrm{FT}^{-1}(\mathrm{FT}(f)\mathrm{FT}(g))$.[32] Alternatively we could try to solve the equations in Fourier space directly, for example by using a multidimensional version of Newton's method at a discrete set of values in the Fourier variable. It may in fact be useful to use Newton's method when iterating in whatever form, see e.g. [75]: rather than updating as $Y_M^{(n+1)} = Y_M^{(n)} + \Delta_M^{(n)}$, where $\Delta_M^{(n)}$ denotes the error of the solution at iteration n, we could update in the direction of greatest linear improvement, i.e. as $Y_M^{(n+1)} = Y_M^{(n)} + \xi_M^{(n)}$ where $\xi_M^{(n)}$ solves $(\delta_M^N - \partial \mathrm{RHS}_M(Y^{(n)})/\partial Y_N)\xi_N^{(n)} = \Delta_M^{(n)}$ and $\mathrm{RHS}_M(Y^{(n)})$ denotes the right hand side of the TBA equations at iteration n.

Regarding the technical implementation of these convolutions and sums, on a computer we cannot work with infinitely many Y functions or integrals over the whole real line. This means that in case of infinitely many Y functions we will have to cut them off at some point, and in any case the integrals will need to be done through some discretized finite interval. Regarding this first point, typically the Y functions for bound states fluctuate less and give smaller contributions to the free energy as the bound state size grows. Consider for instance the constant asymptotics of $Y_Q \sim Q(Q+2)$ that we mentioned in section 3.3, meaning that $\log(1 + 1/Y_Q)$ decreases with Q, unless its relative fluctuations grow in Q, which would be odd. So for practical numerical purposes it may suffice to keep only e.g. the first ten Y functions, unless self-consistency checks based on these first ten indicate that the contributions of higher Y functions are not negligible with regard to the desired accuracy. Importantly, we should not simply drop the other Y functions altogether, but rather add for instance the contribution of their constant asymptotics. This brings us to the second point, integrating over a finite interval. Since we need to cut off the integration domain in some fashion, we need to take care of the asymptotics anyway: cutting the integration domain off at a fixed value means we will introduce a boundary error of order of the asymptotic value at the extrema of the external parameter in the convolution.[33] To reduce this error to an

[31] In the case of the free energy for the Bose gas this can be explicitly shown [4], but let us simply assume this is ok in general at least as long as we do not choose our initial Y functions too poorly.

[32] There are nice exercises with solutions illustrating this as part of the 2012 edition of the Mathematica summer school on theoretical physics available on the web [74].

[33] $K(u-v)$ with $u \sim v$ is of order $K(0)$ which is a relevant scale.

acceptable value we can subtract the equation satisfied by the constant asymptotics, i.e. we solve

$$\log Y_j = \log A_j + \log\left(1+\frac{1}{Y_k}\right) / \left(1+\frac{1}{A_k}\right) \star K_{kj} + a_j, \tag{B.4}$$

where A_i denotes the asymptote of Y_i, which here we assumed to solve the TBA equations with $a_i = 0$. If there are constant nonzero asymptotics in the game, subtracting them is also essential if we wish to Fourier transform. Nonzero constants Fourier transform to delta functions which cannot be reliably implemented numerically. Put differently, functions with constant nonzero asymptotics are not square integrable on the line, so cannot be Fourier transformed in the traditional sense. If we subtract the asymptotics, however, we can readily Fourier transform the fluctuations of interest.

The discussion in this appendix applies equally well to simplified TBA equations – nothing referred to the canonical form of eqn. (B.2) – which importantly are typically faster for numerical purposes as they do not involve infinite (large) sums, but nearest neighbour couplings instead. As discussed we need to be careful about the asymptotics we subtract: in contrast to the canonical equations there are many constant solutions of the basic simplified TBA equations, and we have to choose the one appropriate for our physical situation.

References

[1] N. Andrei and J. Lowenstein, *Diagonalization of the chiral invariant Gross-Neveu Hamiltonian*, *Phys.Rev.Lett.* **43** (1979) 1698.

[2] A. A. Belavin, *Exact solution of the two-dimensional model with asymptotic freedom*, *Phys. Lett.* **B87** (1979) 117–121.

[3] A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szécsényi, and S. J. van Tongeren, eds., *An Integrability Primer for the Gauge-Gravity Correspondence: an Introduction.* Journal of Physics A, 2016. to be published.

[4] C. Yang and C. Yang, *Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction*, *J.Math.Phys.* **10** (1969) 1115–1122.

[5] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. 1. The General solution and the ground state*, *Phys. Rev.* **130** (1963) 1605–1616.

[6] M. Gaudin, *Thermodynamics of the Heisenberg Ising ring for $\Delta \geq 1$*, *Phys.Rev.Lett.* **26** (1971) 1301.

[7] M. Takahashi and M. Suzuki, *A reply to the comments of Johnson et al. on the thermodynamics of the Heisenberg-Ising ring for $|\Delta| < 1$*, *Phys. Lett.* **41A** (1972) 81–82.

[8] M. Takahashi, *Thermodynamics of one-dimensional solvable models.* Cambridge University Press, Cambridge, United Kingdom, 1999.

[9] M. Takahashi, *One-Dimensional Hubbard Model at Finite Temperature*, *Progress of Theoretical Physics* **47** (1972) 69–82.

[10] F. H. L. Essler, H. Frahm, F. Goehmann, A. Kluemper, and V. E. Korepin, *The One-Dimensional Hubbard Model*, *Cambridge University Press, Cambridge, United Kingdom* (2005).

[11] F. Gohmann, A. Klumper, and A. Seel, *Integral representations for correlation functions of the XXZ chain at finite temperature*, *J. Phys.* **A37** (2004) 7625–7652, [hep-th/0405089].

[12] A. B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-Yang Models*, *Nucl. Phys.* **B342** (1990) 695–720.

[13] P. Dorey and R. Tateo, *Excited states by analytic continuation of TBA equations*, *Nucl.Phys.* **B482** (1996) 639–659, [hep-th/9607167].

[14] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, *Integrable quantum field theories in finite volume: Excited state energies*, *Nucl.Phys.* **B489** (1997) 487–531, [hep-th/9607099].

[15] A. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, *Phys.Lett.* **B253** (1991) 391–394.

[16] S. Negro, *Lectures on Integrable Structures in Quantum Field Theory and Massive ODE/IM Correspondence*, in *An integrability primer for the gauge-gravity correspondence* (A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szécsényi, and S. J. van Tongeren, eds.). Journal of Physics A, 2016. arXiv:1606.02952. to be published.

[17] G. Arutyunov and S. Frolov, *Foundations of the* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *Superstring. Part I*, *J.Phys.* **A42** (2009) 254003, [arXiv:0901.4937].

[18] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, et al., *Review of AdS/CFT Integrability: An Overview*, *Lett.Math.Phys.* **99** (2012) 3–32, [arXiv:1012.3982].

[19] G. Arutyunov and S. Frolov, *On String S-matrix, Bound States and TBA*, *JHEP* **12** (2007) 024, [0710.1568].

[20] G. Arutyunov and S. Frolov, *String hypothesis for the* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *mirror*, *JHEP* **03** (2009) 152, [arXiv:0901.1417].

[21] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *Mirror Model*, *JHEP* **05** (2009) 068, [arXiv:0903.0141].

[22] D. Bombardelli, D. Fioravanti, and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: A Proposal*, *J.Phys.A* **A42** (2009) 375401, [arXiv:0902.3930].

[23] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states*, *Lett.Math.Phys.* **91** (2010) 265–287, [arXiv:0902.4458].

[24] J. Ambjorn, R. A. Janik, and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain / string duality*, *Nucl. Phys.* **B736** (2006) 288–301, [hep-th/0510171].

[25] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*, *Phys.Rev.Lett.* **103** (2009) 131601, [arXiv:0901.3753].

[26] A. Cavaglia, D. Fioravanti, and R. Tateo, *Extended Y-system for the AdS_5/CFT_4 correspondence*, *Nucl.Phys.* **B843** (2011) 302–343, [arXiv:1005.3016].

[27] A. Cavaglia, D. Fioravanti, M. Mattelliano, and R. Tateo, *On the AdS_5/CFT_4 TBA and its analytic properties*, arXiv:1103.0499.

[28] J. Balog and A. Hegedus, $\mathrm{AdS}_5 \times \mathrm{S}^5$ *mirror TBA equations from Y-system and discontinuity relations*, *JHEP* **1108** (2011) 095, [arXiv:1104.4054].

[29] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, *Quantum Spectral Curve for Planar $\mathcal{N}$ = Super-Yang-Mills Theory*, *Phys.Rev.Lett.* **112** (2014), no. 1 011602, [arXiv:1305.1939].

[30] Z. Bajnok, A. Hegedus, R. A. Janik, and T. Lukowski, *Five loop Konishi from AdS/CFT*, *Nucl.Phys.* **B827** (2010) 426–456, [arXiv:0906.4062].

[31] G. Arutyunov, S. Frolov, and R. Suzuki, *Five-loop Konishi from the Mirror TBA*, *JHEP* **1004** (2010) 069, [arXiv:1002.1711].

[32] J. Balog and A. Hegedus, *5-loop Konishi from linearized TBA and the XXX magnet*, *JHEP* **06** (2010) 080, [arXiv:1002.4142].

[33] B. Eden, P. Heslop, G. P. Korchemsky, V. A. Smirnov, and E. Sokatchev, *Five-loop Konishi in N=4 SYM*, *Nucl.Phys.* **B862** (2012) 123–166, [arXiv:1202.5733].

[34] N. Drukker, *Integrable Wilson loops*, *JHEP* **1310** (2013) 135, [arXiv:1203.1617].

[35] D. Correa, J. Maldacena, and A. Sever, *The quark anti-quark potential and the cusp anomalous dimension from a TBA equation*, *JHEP* **1208** (2012) 134, [arXiv:1203.1913].

[36] N. Gromov and F. Levkovich-Maslyuk, *Y-system and β-deformed N=4 Super-Yang-Mills*, *J.Phys.A* **A44** (2011) 015402, [arXiv:1006.5438].

[37] G. Arutyunov, M. de Leeuw, and S. J. van Tongeren, *Twisting the Mirror TBA*, *JHEP* **1102** (2011) 025, [arXiv:1009.4118].

[38] D. Bombardelli, D. Fioravanti, and R. Tateo, *TBA and Y-system for planar AdS(4)/CFT(3)*, *Nucl. Phys.* **B834** (2010) 543–561, [arXiv:0912.4715].

[39] N. Gromov and F. Levkovich-Maslyuk, *Y-system, TBA and Quasi-Classical strings in AdS(4) x CP3*, *JHEP* **06** (2010) 088, [arXiv:0912.4911].

[40] F. Levkovich-Maslyuk, *Lectures on the Bethe Ansatz*, in *An integrability primer for the gauge-gravity correspondence* (A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szécsényi, and S. J. van Tongeren, eds.). Journal of Physics A, 2016. arXiv:1605.02950. to be published.

[41] S. J. van Tongeren, *Integrability of the* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *superstring and its deformations*, *J.Phys.* **A47** (2014), no. 43 433001, [arXiv:1310.4854].

[42] H. Saleur and B. Wehefritz-Kaufmann, *Thermodynamics of the complex SU(3) Toda theory*, *Phys.Lett.* **B481** (2000) 419–426, [hep-th/0003217].

[43] G. Arutyunov, M. de Leeuw, and S. J. van Tongeren, *The Quantum Deformed Mirror TBA I*, *JHEP* **1210** (2012) 090, [arXiv:1208.3478].

[44] F. Woynarovich, *On the S(Z)=0 excited states of an anisotropic Heisenberg chain*, *J.Phys.* **A15** (1982) 2985.

[45] F. Woynarovich, *On the eigenstates of a heisenberg chain with complex wavenumbers not forming strings*, *Journal of Physics C: Solid State Physics* **15** (1982), no. 31 6397.

[46] O. Babelon, H. de Vega, and C. Viallet, *Analysis of the Bethe ansatz equations of the XXZ model*, *Nucl.Phys.* **B220** (1983) 13–34.

[47] A. M. Tsvelick and P. B. Wiegmann, *Exact results in the theory of magnetic alloys*, *Advances in Physics* **32** (Nov., 1983) 453–713.

[48] D. Bombardelli, *Lectures on S-Matrices and Integrability*, in *An integrability primer for the gauge-gravity correspondence* (A. Cagnazzo, R. Frassek, A. Sfondrini, I. Szécsényi, and S. J. van Tongeren, eds.). Journal of Physics A, 2016. arXiv:1606.02949. to be published.

[49] L. D. Faddeev, *How algebraic bethe ansatz works for integrable model*, hep-th/9605187.

[50] A. B. Zamolodchikov and A. B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Annals Phys.* **120** (1979) 253–291.

[51] P. Dorey, *Exact S matrices*, hep-th/9810026.

[52] R. A. Janik and T. Lukowski, *From nesting to dressing*, *Phys.Rev.* **D78** (2008) 066018, [arXiv:0804.4295].

[53] A. Kuniba, T. Nakanishi, and J. Suzuki, *T-systems and Y-systems in integrable systems*, *J.Phys.* **A44** (2011) 103001, [arXiv:1010.1344].

[54] G. Arutyunov and S. J. van Tongeren, *The* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *mirror model as a string*, *Phys.Rev.Lett.* **113** (2014) 261605, [arXiv:1406.2304].

[55] G. Arutyunov and S. J. van Tongeren, *Double Wick rotating Green-Schwarz strings*, *JHEP* **1505** (2015) 027, [arXiv:1412.5137].

[56] G. Arutyunov, M. de Leeuw, and S. J. van Tongeren, *The exact spectrum and mirror duality of the* $(\mathrm{AdS}_5 \times \mathrm{S}^5)_\eta$ *superstring*, *Theor.Math.Phys.* **182** (2015), no. 1 23–51, [arXiv:1403.6104].

[57] C. M. Bender and T. T. Wu, *Anharmonic oscillator*, *Phys.Rev.* **184** (1969) 1231–1260.

[58] P. Dorey, *Exact finite size effects in relativistic field theories*, *IGST* (2008) slides can be found at http://web.science.uu.nl/IGST08/pdf/Dorey.pdf.

[59] G. Arutyunov, M. de Leeuw, and S. J. van Tongeren, *The Quantum Deformed Mirror TBA II*, *JHEP* **1302** (2013) 012, [arXiv:1210.8185].

[60] A. Klümper and P. Pearce, *Conformal weights of rsos lattice models and their fusion hierarchiess*, *Physica* **A183** (1992) 304.

[61] A. Klümper and P. Pearce, *New results for exactly solvable critical rsos models and vertex models in two dimensions*, *Physica A: Statistical Mechanics and its Applications* **194** (1993), no. 1 397–405.

[62] D. Fioravanti, A. Mariottini, E. Quattrini, and F. Ravanini, *Excited state Destri-De Vega equation for Sine-Gordon and restricted Sine-Gordon models*, *Phys.Lett.* **B390** (1997) 243–251, [hep-th/9608091].

[63] P. A. Pearce and A. Klümper, *Finite-size corrections and scaling dimensions of solvable lattice models: An analytic method*, *Phys. Rev. Lett.* **66** (1991) 974–977.

[64] A. Klümper and P. A. Pearce, *Analytic calculation of scaling dimensions: Tricritical hard squares and critical hard hexagons*, *Journal of Statistical Physics* **64** (1991) 13–76.

[65] Klümper, A., *Thermodynamics of the Anisotropic Spin-1/2 Heisenberg Chain and Related Quantum Chains*, *Z.Phys.* **B91** (1993) 507, [cond-mat/9306019].

[66] C. Destri and H. de Vega, *New thermodynamic Bethe ansatz equations without strings*, *Phys.Rev.Lett.* **69** (1992) 2313–2317.

[67] C. Destri and H. De Vega, *Unified approach to thermodynamic Bethe Ansatz and finite size corrections for lattice models and field theories*, *Nucl.Phys.* **B438** (1995) 413–454, [hep-th/9407117].

[68] G. Juttner, A. Klümper, and J. Suzuki, *From fusion hierarchy to excited state TBA*, *Nucl.Phys.* **B512** (1998) 581–600, [hep-th/9707074].

[69] A. Kuniba, K. Sakai, and J. Suzuki, *Continued fraction TBA and functional relations in XXZ model at root of unity*, *Nuclear Physics B* **525** (1998) 597–626, [math/98030].

[70] M. Luscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States*, *Commun. Math. Phys.* **104** (1986) 177.

[71] Z. Bajnok and R. A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys.* **B807** (2009) 625–650, [arXiv:0807.0399].

[72] G. Arutyunov, S. Frolov, and S. J. van Tongeren, *Bound States in the Mirror TBA*, *JHEP* **1202** (2012) 014, [arXiv:1111.0564].

[73] R. Benichou, *First-principles derivation of the AdS/CFT Y-systems*, *JHEP* **1110** (2011) 112, [arXiv:1108.4927].

[74] R. Janik, *Day 7, Finite Size Spectrum*, *Mathematica summer school on Theoretical Physics* (2012) exercise sheet and notebook at http://msstp.org/?q=node/277.

[75] M. Takahashi and M. Shiroishi, *TBA equations of 1D hubbard model and high-temperature expansion*, cond-mat/0110645.

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Lectures on Integrable Structures in Quantum Field Theory and Massive ODE/IM Correspondence

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Abstract

This review was born as notes for a lecture given at the YRIS school on integrability in Durham, in the summer of 2015. It deals with a beautiful method, developed in the mid-nineties by V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov and, as such, called BLZ. This method can be interpreted as a field theory version of the quantum inverse scattering (QIS), also known as algebraic Bethe ansatz (ABA). Starting with the case of conformal field theories (CFT) we show how to build the field theory analogues of commuting transfer T matrices and Baxter Q-operators of integrable lattice models. These objects contain the complete information of the integrable structure of the theory, *viz.* the integrals of motion, and can be used, as we will show, to derive the thermodynamic Bethe ansatz (TBA) and non-linear integral (NLIE) equations. This same method can be easily extended to the description of integrable structures of certain particular massive deformations of CFTs; these, in turn, can be described as quantum group reductions of the quantum sine-Gordon model and it is an easy step to include this last theory in the framework of BLZ approach. Finally we show an interesting and surprising connection of the BLZ structures with classical objects emerging from the study of classical integrable models via the inverse scattering transform method. This connection goes under the name of ODE/IM correspondence and we will present it for the specific case of quantum sine-Gordon model only.

This review is part of a collection of papers [33, 38, 35, 26, 13] all of which were born out of lectures given at YRIS school in Durham.

Contents

1 Introduction

The history of Integrable Systems is as old as that of Classical Mechanics and the two were, for the largest part of 18th century, more or less coinciding. Following the formulation of Isaac Newton's laws of motion, for more than a century, eminent mathematicians and physicists such as J. B. d'Alembert, L. Euler, J. L. Lagrange, C. G. J. Jacobi and sir W. R. Hamilton devoted many works to the problem of finding exact solutions to Newton's equations. These efforts brought about a striking amount of results, which condensed in the theory of Lagrangian mechanics first and that of Hamiltonian mechanics then, culminating in the first definition of integrability, as given by J. E. É. Bour and J. Liouville. Although the discovery of many new integrable systems quickly followed, by the end of the 19th century a fundamental result of J. H. Poincaré doused the excitement of the mathematical and physical community, effectively deeming the integrable systems as exceptions amongst the Hamiltonian ones. From that moment the theory of integrable system laid more or less dormant until the second half of the '60, when the idea of integrability resurfaced thanks to the efforts of C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura, P. D. Lax, L. D. Faddeev, V. E. Zakharov and many other. From that pivotal half-decade, the theory of integrable systems begun growing more and more, incorporating results obtained in other branches of theoretical physics, first and foremost Bethe's method for the study of quantum spin chains as well as Baxter's approach to 2D statistical lattice models. The number of publications devoted to the study of integrability grew steadily, especially after the introduction of Conformal Field Theories by A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov and the "first superstring revolution". Finally a last breakthrough came just before the end of the century, thanks to J. Maldacena: the "AdS-CFT" correspondence. This discovery "opened the floodgates" (to borrow the words of Polyakov) and stimulated an impressive amount of work, especially in the field of integrable models.

As the title explicitly reveals, this review deals with the analysis of the integrable structures in field theories. What is meant with this denomination are not simply the fundamental objects that are seen appearing in all integrable theories: the integrals of motion and their "generating functions", the T and Q operators. The expression "integrable structures" encompasses the whole algebraic skeleton which allows for the building of integrability to stand. The study of integrable structures in field theory was first exhaustively addressed to by V.V. Bazhanov, S.L. Lukyanov and

Central charge	Conformal dimension	Spectral parameter
c	h	λ
$\beta=\sqrt{\frac{1-c}{24}}-\sqrt{\frac{25-c}{24}}$	$p\ :\ h=\left(\frac{p}{\beta}\right)^2+\frac{c-1}{24}$	$\theta=(1+\xi)\log(\lambda)$
$q=e^{\mathrm{i}\pi\beta^2}=-e^{-\frac{\mathrm{i}\pi}{1+\xi}}$		$y=\frac{\Gamma\left(1-\beta^2\right)}{\beta^2}\lambda$
$\xi=\frac{\beta^2}{1-\beta^2}$		$\varkappa=-\mathrm{i}\frac{\pi}{\sin(\pi\beta^2)}\lambda$

Tab. 1: Relationship amongst parameters

A.B. Zamolodchikov in a remarkable series of papers [1, 2, 3, 4] and the goal of these 50 odd pages is essentially to go through their work and present it in a uniform and coherent way, providing details and insights in the definitions that we hope will help the reader to understand this beautiful approach. It is, nonetheless, impossible to include in a single review all the implications of the method introduced by Bazhanov, Lukyanov and Zamolodchikov (hereafter addressed to as the BLZ method), as his connections with the theory of quantum integrable systems and CFTs are deep and widespread. For this reason we included a list of references which will be addressed to in the text when a certain topic will be simply cited.

Given the length and the complexity of the subject we decided to keep the main body of the notes as clean as possible by separating heavier calculations and in-depth analyses, not strictly essential to the progression of the review, to boxed sections. These "in-depth boxes" are interleaved with the main body and the reader can, depending on its necessities and the level of its knowledge, skip them without missing anything fundamental about the method. We believe, however, that they can be extremely useful in getting a deeper understanding of the subject and of its many relations to other topics of integrability.

We would like to address a final word of caution to students and young researchers first approaching this subject: do not feel discouraged if you cannot grasp every aspect presented here. The BLZ method requires the use of diverse advanced mathematical concepts and the computations sketched here are often very technical and demanding; insisting on understanding everything at a first reading would be foolish. Instead we suggest multiple readings so that, at each step, it would be possible to go through the concepts, references and calculations in more and more detail. We especially suggest the readers willing to spend time learning this method to go, at some point, through the computations outlined here as this will often force them to explore the references and think about the very meaning of the objects into play: the reward will surely be a deeper and broader understanding of the concepts exposed here.

This review is organised as follows. In the first section, after a brief review of Conformal Field Theories (CFTs) and of classical KdV hierarchy, we will begin building the integrability objects for the $c<1$ CFTs from scratch. We first introduce the quantum transfer matrices $\mathbf{T}_j$ and show how they can be interpreted as sort of generating functions for the quantum integrals of motion. We will then broaden our view, generalising the algebraic setting and constructing the Baxter operators $\mathbf{Q}$. While doing so we will also show how these objects can be used to derive the useful TBA, Bethe Ansatz and NLIE equations, making a connection to the other reviews in this volume. Following this will be a section devoted to the extension of the previous results to the massive integrable deformations of CFTs. This section contains a brief account on the theory of integrable CFT deformations which can be skipped by the readers already familiar with the concept. Finally, in the last section we present a completely different yet, we believe, really interesting approach to the construction of the integrable structures in the particular case of sine-Gordon model. This method, bearing the name ODE/IM correspondence, reveal an intimate and still poorly understood connection between the theory of classical and quantum integrable models. Finally, given the large number of parameters appearing in this review, we thought it would be useful to collect the most relevant ones and the relationship amongst them here, in Table 1.

2 Integrable Structures of Conformal Field Theory

The 2D CFTs are the perfect and probably best known example of exactly solvable quantum field theories. From the year 1984, when the concept of CFT was first introduced in an article of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [6], up to our days they received a great deal of attention and most of their features are now known, to the point of making them a self-contained theory which is very often subject of advanced courses in theoretical physics. The usual approach of these courses concerns what we might call a representation-theoretical

characterisation of CFTs, that is to say a description and classification of their spectrum in terms of modules over the Virasoro algebra (or one of its extensions). This point of view, which employs a wide array of mathematical concepts, ranging from representation of infinite-dimensional algebras to number theory, has the advantage of being extremely neat and crystalline clear; however the concepts directly related to integrability as we know them from classical and lattice models, such as integrals of motion, Bethe ansatz equations and so on, do not seem to play a primary role and remain hidden somewhere inside this elegant construction. Such is its power that the legitimate question whether there is actually any need to address to the integrable structures in CFTs arises. Anyhow, this "conventional" point of view is limited to CFTs only and, when dealing with, for example, their deformations one would like methods closer to those employed in their lattice regularisation, or in their classical limits, to be available. Looking in this direction, there exists an alternative approach to CFTs, pioneered by Al.B.Zamolodchikov [7], where the spectrum is described in terms of scattering states of a set of massless particles[1]. The fundamental object in this context is the factorisable *massless S-matrix*. This approach is closer to the integrable structures, *viz.* it is closer to a separated variables description. The aim of this first part is to describe this integrable structure of CFT by building the "fundamental objects": the operators $\mathbf{T}_j$ and $\mathbf{Q}_\pm$. These can be interpreted as the continuum analogues of commuting transfer matrices and Baxter operators of integrable lattice models and, in fact, it is wise to keep in mind this parallel with lattice models, in order to understand the origin and meaning of most formulae.

2.1 Brief overview of CFT basic concepts

The goal of the following section is to recall the basics of 2D CFT and to set up the notation, without any pretense of completeness. It is intended for readers having already a good knowledge of the subject; for those who are less familiar with it, there exists a plethora of, often very good, reviews and books dealing with 2D CFT, offering a wide range of different points of view. I suggest [8] for a straightforward introduction and the references therein for a more in-depth study; the most daring might consider the "Big Yellow Book" by P. Di Francesco, P. Mathieu and D. Sénéchal [9].

The Virasoro algebra A Conformal Field Theory (CFT) in D Euclidean dimensions, that is a local, isotropic field theory possessing no characteristic length scale, is invariant under the global conformal group $SO(D+1,1)$, a non-compact Lie group of dimension $\frac{1}{2}(D+1)(D+2)$. Although bigger than the Galilei group $\mathbb{R}^D \rtimes SO(D)$, whose dimension is $\frac{1}{2}D(D+1)$, it is still not sufficient to grant integrability to the CFT: we need an infinite number of symmetries to perform this task[2] and that is exactly what we find if we look at the particular case $D=2$. In fact when we consider the conformal transformations of a plane, even if the special orthogonal group $SO(3,1)$ has dimension 6, we can find an infinite number of conformal coordinate transformations: the holomorphic mappings from the complex plane (or part of it) onto itself. This discrepancy between the finiteness of the conformal group and the infinite amount of independent conformal coordinate transformations is easily resolved by remarking that most of these last are not globally well-defined. The set of infinitesimal conformal transformations form an infinite dimensional local algebra, the renowned *Virasoro algebra*[3] *Vir*, which exponentiate to the *Virasoro group* $\widehat{\mathrm{diff}S^1}$ (the centrally extended group of diffeomorphisms of the unit circle) [10, 11]. This last contains, as a subgroup, the Möbius group $SL(2,\mathbb{C})/\mathbb{Z}_2 \sim SO(3,1)$ of global conformal transformations. Since a local field theory should be sensitive to local symmetries, even if the related transformations are not globally defined, the behaviour of a $(1+1)$-dimensional is indeed constrained by the full algebra Vir. It is precisely the local conformal invariance that, being infinite-dimensional, allows for exact solutions of 2D CFTs to exist. The Virasoro algebra *Vir* is generated by the operators $\{L_n\}_{n\in\mathbb{Z}}$, obeying the famous commutation relations

$$[L_n,\, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n\,(n^2-1)\,\delta_{n,-m}\ ,$$

[1] Note that this characterisation is not unique! One can choose different massless particle bases of the Hilbert space, with different particle content and different scattering amplitudes. These different choices reflect the possibility of reaching a certain CFT as massless limit of different massive field theories.

[2] The intuitive reason for that comes from Liouville's definition of an integrable system. This states that a system is integrable iff it possesses the same number of conserved quantities and degrees of freedom. By Nöther theorem conserved charges correspond to symmetries and so we need a set of symmetries having dimension equal to the degrees of freedom of our system. But a field theory has "infinite degrees of freedom"! Hence we need an infinite number of symmetries to make it integrable.

[3] Actually the algebra of infinitesimal conformal mappings is the Witt algebra. However in the quantum theory this symmetry is anomalous and the Witt algebra gets extended to *Vir* by the addition of a *central charge* c, also called *conformal anomaly*.

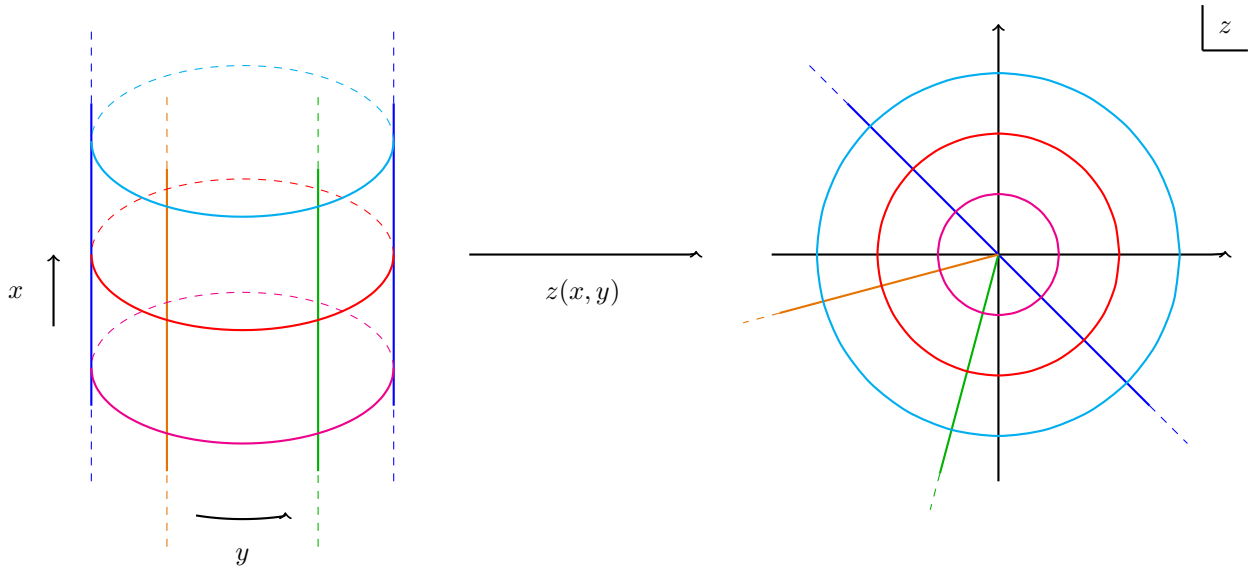


Fig. 2.1: Map from the cylinder (x, y) to the complex z-plane.

where c is a number called central charge or conformal anomaly.

Let us now consider a CFT on a flat Euclidean plane, having coordinates (x, y). The presence of scale invariance means that we are dealing with massless theories. Were we considering a Minkowski geometry (x^0, x^1), it would be then natural to describe the system using light-cone coordinates $(x^+, x^-) = (x^1 + x^0, x^1 - x^0)$, so that left/right-moving massless fields depend uniquely on, respectively, x^+ or x^-. Here, in the same spirit, it turns out to be extremely useful to introduce the complex coordinates $(w, \overline{w}) = (x + \mathrm{i}y, x - \mathrm{i}y)$ and the notion of left/right-moving fields turns into that of purely holomorphic/antiholomorphic Euclidean fields. The algebra of symmetries of a CFT will thus be the direct sum of two Virasoro algebras: $Vir \oplus \overline{Vir}$. From now on we will consider the complex coordinates w and $\bar{w}$ to be independent, so that $Vir \oplus \overline{Vir}$ naturally acts on $\mathbb{C}^2$ and we can treat each term in the direct sum independently and effectively work only with holomorphic fields. When the time comes to compute physical quantities we will "remember" to add the antiholomorphic contributions and impose the "real slice" condition $\bar{w} = w^*$.

Since we are dealing with massless fields, we must pay attention to infrared divergencies. For this reason we invest one of the dimensions, say y, with the rôle of spatial dimension and compactify it on a unit circle: $y + 2\pi \equiv y$. This procedure defines our theory on a cylinder $\mathbb{R} \times S^1$. Next we can perform the following conformal map

$$w \longrightarrow z \doteq e^w \equiv e^{x+\mathrm{i}y} \,,$$

which "squashes" the cylinder on the z-complex plane as shown pictorially in Figure 2.1. It is easy to see that the "time" direction x is mapped in the radial one $\rho = \sqrt{z\overline{z}}$, while the "space" direction y is sent in the angular one $\phi = \frac{1}{2\mathrm{i}} \log\left(\frac{z}{\overline{z}}\right)$. The infinite past and infinite future $x = \pm\infty$ are sent to the points 0 and ∞, respectively, of the z-plane. The usual procedure of quantisation in this setup is called *radial quantisation*.

The subalgebra of $Vir \times \overline{Vir}$ generated by $\left\{L_i, \overline{L}_i\right\}_{i=-1}^{1}$ is associated with the global conformal group $SL(2, \mathbb{C})/\mathbb{Z}_2$ and is anomaly free. It is useful for characterising physical states. In fact suppose that we are working, as we will, with eigenstates of the operators L_0 and $\bar{L}_0$ and denote the eigenvalues, called *conformal weights, as* h and $\bar{h}$. Consider the following two particular operators of $SL(2, \mathbb{C})/\mathbb{Z}_2$:

- $L_0 + \overline{L}_0$: on the cylinder it generates the translations along the time direction and gets mapped, on the plane, to the generator of dilatations $(z, \overline{z}) \to \gamma(z, \overline{z})$. In radial quantisation, it corresponds to the Hamiltonian of the system;
- $\mathrm{i}(L_0 - \overline{L}_0)$: on the cylinder it generates the translations along the space direction and gets mapped, on the plane, to the generator of rotations $(z, \overline{z}) \to (e^{\mathrm{i}\alpha} z, e^{-\mathrm{i}\alpha} \overline{z})$. In radial quantisation, it corresponds to the momentum of the system.

The eigenvalues of these two operators are the *scaling dimension* $\Delta \doteq h + \bar{h}$ and the *conformal spin* $s \doteq h - \bar{h}$. In the context of radial quantisation they correspond to the energy and the momentum of the state.

The energy-momentum tensor The energy-momentum tensor $T^{\mu\,\nu}$ is defined as the conserved current associated to the invariance of the system with respect to coordinate transformations $\epsilon_\mu(x)$:

$$\delta_\epsilon \mathscr{A} = \frac{1}{2}\int d^2x\, T^{\mu\,\nu}(x)\ (\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x))\ ,$$

where $\mathscr{A}$ is the action of the system. The energy-momentum tensor is symmetric in its indices $T^{\mu\,\nu} = T^{\nu\,\mu}$ and for a CFT it is traceless $T^\mu_\mu = 0$ (this is actually valid in any dimension D). If $D = 2$, then, we only have one component for each chirality:

$$T_{\text{plane}}(z) \doteq -2\pi T^{z\,z}\ , \qquad \overline{T}_{\text{plane}}(\overline{z}) \doteq -2\pi T^{\overline{z}\,\overline{z}}\ .$$

These two components are the generating functions of the Virasoro generators

$$T_{\text{plane}}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\ , \qquad \overline{T}_{\text{plane}}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{z}^{-n-2}\overline{L}_n\ ,$$

which, in turn, can be expressed in terms of T by means of Cauchy theorem:

$$L_n = \frac{1}{2\pi \mathsf{i}}\oint_0 dz\, z^{n+1} T_{\text{plane}}(z)\ , \qquad \overline{L}_n = \frac{1}{2\pi \mathsf{i}}\oint_0 d\overline{z}\, \overline{z}^{n+1}\overline{T}_{\text{plane}}(\overline{z})\ .$$

These formulae allow to build the energy-momentum tensor also for those CFT who do not possess an action (or if that action is not known). In the following we will be considering CFTs defined on the cylinder and is useful to have an expression of the energy-momentum tensor in this geometry:

$$T(w) = -\frac{c}{24} + \sum_{n\in\mathbb{Z}} e^{i\,n\,w} L_{-n}\ , \qquad \overline{T}(\overline{w}) = -\frac{c}{24} + \sum_{n\in\mathbb{Z}} e^{-i\,n\,\overline{w}} L_{-n}\ .$$

The corresponding expressions of the Virasoro generators take the form

$$L_n = \frac{1}{2\pi}\int_0^{2\pi} dw\, e^{inw} T(w)\ , \qquad \overline{L}_n = \frac{1}{2\pi}\int_0^{2\pi} d\overline{w}\, e^{-in\overline{w}}\overline{T}(\overline{w})\ .$$

It is useful to remark here that on the cylinder $T(w + 2\pi) = T(w)$ and $\overline{T}(\overline{w} + 2\pi) = \overline{T}(\overline{w})$.

Primary fields Let us consider a transformation of coordinates $(z, \overline{z}) \to (\omega(z), \overline{\omega}(\overline{z}))$; any field in our CFT which transforms as follows

$$\varphi_{h,\overline{h}}(z,\overline{z}) \to \varphi'_{h,\overline{h}}(\omega(z),\overline{\omega}(\overline{z})) = \left(\frac{d\omega}{dz}\right)^{-h}\left(\frac{d\overline{\omega}}{d\overline{z}}\right)^{-\overline{h}} \varphi_{h,\overline{h}}(z,\overline{z})\ ,$$

is named *primary* field, while the quantities h and $\overline{h}$ are the holomorphic and anti-holomorphic conformal weights introduced above. All the fields which are not primary will be called *descendants*. The energy-momentum tensor is an example of a particular descendant field, called *quasi-primary* as it transforms as a primary only under global conformal transformations:

$$T(z) \to T'(\omega(z)) = \left(\frac{d\omega}{dz}\right)^{-h} T(z) + \frac{c}{12}\{z;\omega(z)\}\ ,$$

where $\{z;\omega(z)\}$ is the Schwartzian derivative of $\omega(z)$, which vanishes iff $\omega \in SL(2,\mathbb{C})/\mathbb{Z}_2$. A consequence of the form of primary fields and energy-momentum tensor is the particular operator product expansion (OPE) that they satisfy:

$$T(z)\varphi_{h,\overline{h}}(z',\overline{z}') \sim \frac{h}{(z-z')^2}\varphi_{h,\overline{h}}(z',\overline{z}') + \frac{1}{z-z'}\partial_{z'}\varphi_{h,\overline{h}}(z',\overline{z}')\ ,$$

$$T(z)T(z') \sim \frac{c/2}{(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}\partial_{z'}T(z')\ .$$

Hilbert space The Hilbert space $\mathscr{H}_{ph}$ of our CFT is built on some vacuum state $|\,0\,\rangle$, which must be invariant under $SL(2,\mathbb{C})/\mathbb{Z}_2$ and satisfy

$$L_n\,|\,0\,\rangle = 0\;, \qquad \overline{L}_n\,|\,0\,\rangle = 0\;, \quad \forall n \geq -1\;,$$

which is a consequence of the request that $T(z)\,|\,0\,\rangle$ and $\overline{T}(\overline{z})\,|\,0\,\rangle$ be well-defined as $(z,\overline{z}) \to (0,0)$[4]. The action of a primary field on this vacuum generates eigenstates of the Hamiltonian:

$$\big|\,h,\overline{h}\,\big\rangle \doteq \varphi_{h,\overline{h}}(0,0)\,|\,0\,\rangle\;.$$

The fact that these are eigenstates is easily obtained from the OPE properties of the primary fields:

$$[L_0, \varphi_{h,\overline{h}}(0,0)] = \frac{1}{2\pi\mathrm{i}}\oint_0 dz\, z\, T(z)\,\varphi_{h,\overline{h}}(0,0) = h\varphi_{h,\overline{h}}(0,0)\;.$$

Similar properties are valid for other operators L_n and one finds

$$\begin{matrix} L_0\,\big|\,h,\overline{h}\,\big\rangle = h\,\big|\,h,\overline{h}\,\big\rangle \\ \overline{L}_0\,\big|\,h,\overline{h}\,\big\rangle = \overline{h}\,\big|\,h,\overline{h}\,\big\rangle \end{matrix} \quad ; \quad \begin{matrix} L_n\,\big|\,h,\overline{h}\,\big\rangle = 0 \\ \overline{L}_n\,\big|\,h,\overline{h}\,\big\rangle = 0 \end{matrix} \quad , \quad \forall n > 0\;.$$

So these states, which we call primary like the fields generating them, are highest-weight vectors for $\mathit{Vir} \times \overline{\mathit{Vir}}$. We can thus generate a subset $\mathcal{V}_h \otimes \mathcal{V}_{\overline{h}}$ of the Hilbert space by the free action of $\big\{L_n, \overline{L}_n\big\}_{n=-\infty}^{-1}$ on the primary state $\big|\,h,\overline{h}\,\big\rangle$:

$$\mathcal{V}_a \equiv \mathcal{V}_{h_a} \doteq \Big\{ L_{-k_1}L_{-k_2}\ldots L_{-k_n}\,\big|\,h,\overline{h}\,\big\rangle \;\Big\backslash\; 1 \leq k_1 \leq \cdots \leq k_n \;\Big\backslash\; \forall n \geq 0 \Big\}\;.$$

The vector field $\mathcal{V}_a$ is closed under the action of *Vir* and it's called *Verma module*.

The Hilbert space of a CFT is embedded in some suitable way (we will not cover this topic here) into $\mathscr{H}_{ch} \otimes \overline{\mathscr{H}}_{ch}$, where $\mathscr{H}_{ch} \doteq \bigoplus_a \mathcal{V}_a$ is the space of "right-chiral" states. The index a of the direct sum runs on the admissible conformal dimension of our CFT; this number depends on c and is usually infinite. There are however some notable exceptions where the structure of the Verma modules becomes degenerate and the number of allowed highest weights becomes finite. Let us list three important categories of CFTs:

- **Unitary non degenerate**: when $c \geq 1$, the conformal dimension may take a continuum of positive values; for any of these, $\mathcal{V}_h$ is unitary. An example in this class is the free boson, corresponding to $c = 1$;
- **Minimal models**: when

  $$c = 1 - 6\frac{(m-m')^2}{m\,m'}\;, \quad m \perp m'\;,\; m,m' \in \mathbb{N}/\,\{0\}\;,$$

  the allowed conformal dimensions are restricted to take a finite number of discrete values, indexed by two integers (r,s):

  $$h_{r,s} = \frac{(m\,r - m's)^2 - (m-m')^2}{4\,m\,m'}\;, \qquad \begin{matrix} 1 \leq r < m' \\ 1 \leq s < m \end{matrix}\;.$$

  These models are denoted as $\mathcal{M}_{m,m'}$. An example in this class is the Yang-Lee model, corresponding to the choice $m = 5$ and $m' = 2$, meaning $c = -\frac{22}{5}$;
- **Unitary minimal models**: amongst the above models, the unitary ones are those with $m' = m+1$ and they usually are denoted as $\mathcal{M}_m$. An example in this class is the Ising model, corresponding to $m = 3$ and $c = \frac{1}{2}$.

[4] This request contain in itself that of invariance with respect to $SL(2,\mathbb{C})/\mathbb{Z}_2$.

Integrals of motion The first step towards the unveiling of the integrable structures of CFT was performed by R. Sasaki and I. Yamanaka [12]. They considered a CFT on a cylinder and the algebra $\mathcal{U}(Vir)$ generated by the energy-momentum tensor $T(w)$ along with composite fields built as (normal ordered) powers of $T(w)$ and its derivatives. What they found is that there exists an infinite dimensional abelian subalgebra $\mathcal{I} \subset \mathcal{U}(Vir)$ spanned by "local integrals of motion" (IM):

$$\mathcal{I} = \{\mathbf{I}_{2k-1}\}_{k=1}^{\infty} \ , \quad \mathbf{I}_{2k-1} \doteq \int_0^{2\pi} \frac{dw}{2\pi} T_{2k}(w) \ , \quad [I_{2k-1}, I_{2l-1}] = 0 \ ,$$

where the cylinder radius is fixed at $R = 1$. The densities are some regularised polynomials of $T(w)$ and its derivatives; for example

$$T_2(w) \doteq T(w) \ , \quad T_4(w) \doteq :T^2(w): \ , \quad T_6(w) \doteq :T^3(w): + \frac{c+2}{12} :(\partial T(w))^2: \ , \tag{2.1}$$

where the regularised product is defined as

$$:T^2(w): \doteq \oint_w \frac{dw'}{2\pi \mathrm{i}} \frac{\mathscr{T}\left(T(w')T(w)\right)}{w'-w} \ ,$$

and $\mathscr{T}$ is the "chronological product"[5]

$$\mathscr{T}\left(A(w')B(w)\right) = \begin{cases} A(w')B(w) & \text{if } \Im(w) > \Im(w') \\ B(w)A(w') & \text{if } \Im(w') > \Im(w) \end{cases} .$$

Although there is no known closed formula for the densities $T_{2k}(w)$, they are uniquely determined by the requirement of commutativity of local IMs and by the *spin assignment* rule:

$$\oint_w \frac{dw'}{2\pi \mathrm{i}} (w'-w) \mathscr{T}\left(T(w')T_{2k}(w)\right) = 2k\, T_{2k}(w) \ ,$$

which can be simply implemented by requiring $T_{2k}(w)$ to be a polynomial of total grade $2k$ and assigning grade 2 to T and grade 1 to derivatives. The first few IMs are "easily" computed

$$\mathbf{I}_1 = L_0 - \frac{c}{24} \ , \qquad \text{This is the (chiral part of the) Hamiltonian!} \ , \tag{2.2}$$

$$\mathbf{I}_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + c\frac{5c+22}{4\times 6!} \ , \tag{2.3}$$

$$\begin{aligned} \mathbf{I}_5 = {} & \sum_{n_1+n_2+n_3=0} :L_{n_1}L_{n_2}L_{n_3}: + \sum_{n=1}^{\infty}\left[\frac{c+11}{6}n^2 - 1 - \frac{c}{4}\right] L_{-n}L_n + \frac{3}{2}\sum_{n=1}^{\infty} L_{1-2n}L_{2n-1} + \\ & - \frac{c+4}{8}L_0^2 + 5(c+2)\frac{3c+20}{4\times 6!}L_0 - 5c\frac{3c+14}{4\times 9!}(7c+68) \end{aligned} \tag{2.4}$$

2.2 Brief overview of classical KdV

In this section we will briefly present some very basic concepts and facts about the classical KdV hierarchy. The reader interested in this topic can find a good starting point for the study of classical integrability in the review [13]; for a more advanced read we suggest the beautiful book [14].

Why being concerned with the classical KdV? The reason is very simple. It is known [12] that the CFTs with $c < 1$ are, in some sense, a quantum version of the classical KdV; in fact if we consider the "classical limit" $c \to -\infty$ and perform the following substitutions:

$$T(w) \to -\frac{c}{6}U(w) \ , \qquad [\ ,\ ] \to \frac{6\pi}{\mathrm{i}\, c}\{\ ,\ \}_P \ ,$$

[5] This is the corresponding on the cylinder of the radial ordering on the z-plane.

where $\{\ ,\ \}_P$ are the Poisson brackets, the Virasoro algebra reduces to the following Poisson algebra

$$\{U(w), U(w')\}_P = 2\left(U(w) + U(w')\right)\delta'(w-w') + \delta'''(w-w')\ ,$$

which is known to describe the second Hamiltonian structure of KdV, provided the Hamiltonian is chosen amongst the classical IM:

$$\begin{aligned} I_1^{\rm cl} &= \int_0^{2\pi} \frac{dw}{2\pi} U(w) \\ I_3^{\rm cl} &= \int_0^{2\pi} \frac{dw}{2\pi} U^2(w) \\ I_5^{\rm cl} &= \int_0^{2\pi} \frac{dw}{2\pi} \left[U^3(w) - \frac{1}{2}\left(\partial_w U(w)\right)^2\right] \\ &\cdots \quad . \end{aligned}$$

We choose the field $U(w)$ to be periodic $U(w+2\pi) = U(w)$, just like $T(w)$. These classical IMs, which form a commutative Poisson algebra $\{I_{2k-1}, I_{2l-1}\}_P = 0$, are clearly the classical versions of the operators (2.2-2.4). Different choices of Hamiltonian bring us to different equation of motion:

$$\begin{aligned} I_1^{\rm cl} &: \quad \partial_{t_1} U = \partial_w U \\ I_3^{\rm cl} &: \quad \partial_{t_3} U = \partial_w^3 U + 6U\,\partial_w U \qquad \text{The "canonical" KdV} \\ I_5^{\rm cl} &: \quad \partial_{t_5} U = -\partial_w^5 U - 2U\,\partial_w^3 U + 5\partial_w U \partial_w^2 U + 20U^2 \partial_w U \\ &\cdots \quad . \end{aligned} \tag{2.5}$$

This infinite sequence of partial differential equations is called *KdV hierarchy* and it can be shown to be equivalent to a description of the isospectral deformations of the following second order differential operator depending on a *spectral parameter* λ:

$$L(w|\lambda) \doteq \partial_w^2 + U(w) - \lambda^2\ , \tag{2.6}$$

called *Lax operator*. The connection between this operator and the tower of differential equations (2.5) relies on the existence of an infinite set of operators $M_{2n-1}(w)$, such that

$$\frac{d}{dt_{2n-1}} L(w|\lambda) = [M_{2n-1}(w), L(w|\lambda)] \iff \text{KdV equation associated to } I_{2n-1}^{\rm cl} \text{ is satisfied.}$$

For example the canonical KdV equation is obtained from the operator

$$M_3(w) = 4\partial_w^3 + 6U(w)\partial_w + 3U'(w)\ .$$

Associated to each Lax operator, there exists a differential equation, called usually *auxiliary equation* (or *system* if one has to deal with matrix Lax operators). In our case the equation has the form (from here on the prime $'$ will denote differentiation)

$$L(w|\lambda)\psi(w|\lambda) = \psi''(w) - \left(\lambda^2 - U(w)\right)\psi(w) = 0\ .$$

This is a second order differential equation and, as such, possesses two linearly independent solutions $\psi_1(w|\lambda)$ and $\psi_2(w|\lambda)$. Very important characteristics of differential equations are the *monodromy properties* of the solutions; these can be encoded in the *monodromy matrix*, defined as

$$\left(\psi_1(w|\lambda)\,,\ \psi_2(w|\lambda)\right)\mathbf{M}(\lambda) = \left(\psi_1(w+2\pi|\lambda)\,,\ \psi_2(w+2\pi|\lambda)\right)\ .$$

Out of the monodromy matrix is then possible to define the *T-function*. This is a central object in integrable systems and is defined most simply as the trace of $\mathbf{M}$

$$\mathbf{T}(\lambda) \doteq tr\mathbf{M}(\lambda)\ . \tag{2.7}$$

Although an explicit expression of $\mathbf{T}$ can be complicated to obtain, we can express it as an asymptotic series at large λ:

$$\frac{1}{2\pi}\log\left[\mathbf{T}(\lambda)\right] \underset{\lambda\to\infty}{\sim} \lambda\left[1-\sum_{n=1}^{\infty} c_n I^{\mathrm{cl}}_{2n-1}\lambda^{-2n}\right] , \tag{2.8}$$

where $c_1 = 1/2$ and $c_n = \frac{(2n-3)!!}{2^n\, n!}$.

WKB expansion of the Lax auxiliary equation

It is instructive to compute explicitly the expression of the monodromy matrix. In order to do so we need to find a representation of the solutions to the differential equation

$$\psi''(w) = \left(\lambda^2 - U(w)\right)\psi(w) ,$$

and a standard procedure which allows us to do so is the WKB method [15]. The first step consists in introducing a small parameter ϵ^2 in front of the second derivative:

$$\epsilon^2\psi''(w) = \left(\lambda^2 - U(w)\right)\psi(w) ,$$

and search for solutions of the form

$$\psi(w) \sim \exp\left[\frac{1}{\epsilon}S(w) + A_0(w) + \sum_{n=1}^{\infty}\epsilon^n A_n(w)\right] , \text{as } \epsilon\to 0 ,$$

where the sign $\sim$ reminds us that the right-hand side is an asymptotic series. By inserting this form in the differential equation and isolating each power of ϵ, we find

$$\begin{aligned}
\epsilon^0 \;:\quad & S'(w)^2 = \lambda^2 - U(w) \quad\Rightarrow\quad S(w) = \pm\int_{w_0}^{w}\sqrt{\lambda^2 - U(w')}dw' ,\\
\epsilon^1 \;:\quad & S''(w) + 2S'(w)A_0'(w) = 0 \quad\Rightarrow\quad A_0(w) = k - \frac{1}{4}\log\left(\lambda^2 - U(w)\right) ,\\
\epsilon^n \;:\quad & 2S'(w)A_{n-1}'(w) + A_{n-2}''(w) + \sum_{k=0}^{n-2}A_k'(w)A_{n-k}'(w) = 0 , \quad \forall n > 1 .
\end{aligned}$$

This is a triangular system of differential equations, which allows us to obtain the n-th term by the simple integration of a first-order differential equation. What's more, each even order equation happens to be the difference of total derivatives; for example

$$\epsilon^3 \;:\qquad A_2'(w) = \partial_w\left[\frac{\left(\partial_w\log S'(w)\right)^2 - 2\partial_w^2\log S'(w)}{16\, S'(w)^2}\right] .$$

The odd-order equations, on the other hand, are proper first-order differential equations, as an example, the first term reads

$$\epsilon^2 \;:\qquad A_1(w) = \int_{w_0}^{w}\frac{2S'''(w')S'(w') - 3S''(w')^2}{8S'(w')^3}dw' .$$

Now, in order to obtain an expression for $\mathbf{M}$, we need to see what happens to our solution when we shift $w\to w+2\pi$. This is easily computed remembering that $U(w+2\pi) = U(w)$, so that

$$A_{2n}(w+2\pi) = A_{2n}(w) , \qquad \forall n \geq 0 ,$$

while

$$A_{2n-1}(w+2\pi) = \int_{w_0}^{w+2\pi} \mathcal{A}_{2n-1}\left[U(w')\right] dw' = \int_{w_0-2\pi}^{w_0} \mathcal{A}_{2n-1}\left[U(w')\right] dw' + \int_{w_0}^{w} \mathcal{A}_{2n-1}\left[U(w')\right] dw' \ ,$$

so that

$$A_{2n-1}(w+2\pi) = A_{2n-1}(w) + \int_{0}^{2\pi} \mathcal{A}_{2n-1}\left[U(w')\right] dw' \ ,$$

where $\mathcal{A}_{2n-1}\left[U(t)\right]$ is some functional of $U(t)$ and $A_{-1}(w) \equiv S(w)$. From this we easily infer that the monodromy matrix is diagonal with eigenvalues

$$\exp\left[\pm \int_0^{2\pi} \left(\sqrt{\lambda^2 - U(w')} + \sum_{n=1}^{\infty} \mathcal{A}_{2n-1}\left[U(w')\right]\right) dw'\right] \ .$$

The first two terms in the large-λ expansion of $\mathbf{T}$ (2.8) come from the expansion of the square root for large λ,

$$\int_0^{2\pi} \frac{dw'}{2\pi} \sqrt{\lambda^2 - U(w')} \underset{\lambda\to\infty}{\sim} \int_0^{2\pi} \frac{dw'}{2\pi} \lambda \left(1 - \frac{U(w')}{2\lambda^2} - \frac{U(w')^2}{8\lambda^4} \dots\right) = \lambda\left(1 - \frac{1}{2\lambda^2} I_1^{\rm cl} - \frac{1}{8\lambda^4} I_3^{\rm cl}\right) \ .$$

On the other hand, higher order terms require the computation of more and more $A_n(w)$ in the WKB expansion; as a simple example, let us take in consideration A_1

$$\mathcal{A}_1(w) = -\frac{4U''(w)(\lambda^2 - U(w)) - 5U'(w)^2}{32(\lambda^2 - U(w))^{\frac{5}{3}}} \underset{\lambda\to\infty}{\sim} -\frac{U''(w)}{8\lambda^3} - \frac{5U'(w)^2 + 6U(w)U''(w)}{32\lambda^5} \ .$$

When integrating the above term between 0 and 2π, all total derivatives vanish and we can perform integration by parts, so that the coefficient of λ^{-5} reads, as expected

$$-\frac{1}{16}\int_0^{2\pi} \frac{dw'}{2\pi}\left(U(w')^3 - \frac{1}{2}U'(w')^2\right) = -\frac{1}{16} I_5^{\rm cl} \ .$$

We have thus shown that the T-function (2.7) serves as a sort of generating function for the classical IMs (2.8). However we can do more, much more: in fact we can construct an infinite tower of Poisson commuting T-functions! This is a consequence of a deep connection between KdV hierarchy and the Lie algebra $sl(2)$ as we are going to briefly hint at. A close look at the differential operator (2.6) shows that it can be factorised:

$$L(w|\lambda) = \left(\partial_w + \phi'(w)\right)\left(\partial_w - \phi'(w)\right) - \lambda^2 \ ,$$

with the field $\phi(w)$ being the *Miura transform* of $U(w)$ [16]:

$$-U(w) = \left(\phi'(w)\right)^2 + \phi''(w) \ , \tag{2.9}$$

having canonical Poisson brackets

$$\{\phi(w), \phi(w')\}_P = \epsilon(w - w') \ , \qquad \epsilon(x) = n \ , \ 2\pi n < x \leq 2\pi(n+1) \ .$$

Note that since the field $U(w)$ is periodic, the Miura field has to be taken, in full generality, quasiperiodic

$$\phi(w + 2\pi) = \phi(w) + 2\pi i p \ .$$

We can now reduce the second order differential equation $L(w|\lambda)\psi(w|\lambda) = 0$ to a system of first order equations:

$$\begin{cases} (\partial_w - \phi'(w))\,\psi(w|\lambda) = \lambda\tilde{\psi}(w|\lambda) \\ (\partial_w + \phi'(w))\,\tilde{\psi}(w|\lambda) = \lambda\psi(w|\lambda) \end{cases} ,$$

which can be written in matrix form

$$\left(\partial_w - \phi'(w)\sigma^3 - \lambda\sigma^1\right)\Psi(w|\lambda) = 0 \,,$$

with σ^i being the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Now comes the generalisation: since the connection between L and the hierarchy of KdV equations (2.5) uniquely relies on the commutation relations between L and the operators M_{2n-1}, we can think of defining an abstract Lax operator

$$\mathscr{L}(w|\lambda) \doteq \partial_w - \phi'(w)\,H - \lambda(E+F) \,,$$

where H, E and F are the generators of $sl(2)$ Lie algebra:

$$[H,E] = 2E \,, \quad [H,F] = -2F \,, \quad [E,F] = 2H \,.$$

The commutation properties of this operator are exactly the same as those of L and, moreover, it reduces to this last when the 2-dimensional representation of $sl(2)$ is chosen. We thus expect the monodromy properties of this operator to encode information on the KdV hierarchy. Now, let π_j, with $j \in \frac{1}{2}\mathbb{N}$, denote the $(2j+1)$-dimensional representation of $sl(2)$, such that $\pi_j[H] = \mathrm{diag}\,(2j, 2j-2, \ldots, -2j+2, -2j)$. Consider the matrix equation

$$\pi_j\left[\mathscr{L}(w|\lambda)\right]\Psi_j(w|\lambda) = 0 \,,$$

where $\Psi_j(w|\lambda)$ is a $(2j+1)$-dimensional vector, and let us repeat what has been done just above. In order to obtain a nice form of the solution to this equation, we rewrite it as follows (we omit π_j in the next few equations, for clarity):

$$(\partial_w - \phi'(w)H)\,\Psi(w) = e^{\phi(w)H}\partial_w e^{-\phi(w)H}\Psi(w) = \lambda\,(E+F)\,\Psi(w) \,,$$

where the first passage is allowed, since H is diagonal. Now define $\tilde{\Psi}(w) \doteq e^{-\phi(w)H}\Psi(w)$, so that it satisfies the equation

$$\partial_w\tilde{\Psi}(w) = \lambda e^{-\phi(w)H}\,(E+F)\,e^{\phi(w)H}\tilde{\Psi}(w) = \lambda\left(e^{-2\phi(w)}E + e^{2\phi(w)}F\right)\tilde{\Psi}(w) \,,$$

where we used the property of any Lie algebra element A: $e^{\alpha H}Ae^{-\alpha H} = e^{\alpha\mathrm{ad}_H(A)}A$, where the adjoint action is defined by $[H,A] = \mathrm{ad}_H(A)\,A$. The general solution of a first-order matrix equation can be written as a *path-ordered exponential*:

$$\tilde{\Psi}(w) = \mathscr{P}\exp\left[\lambda\int_0^w dw'\left(e^{-2\phi(w')}E + e^{2\phi(w')}F\right)\right]\Psi^0 \,,$$

with Ψ^0 being an arbitrary constant vector, representing the integration constants. A path-ordered exponential $\mathscr{P}\exp\left[\int_0^w a(w')dw'\right]$ is defined as the following series expansion

$$\mathscr{P}\exp\left[\int_0^w a(w')dw'\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^w\cdots\int_0^w \mathscr{P}\left[a(w'_1)\cdots a(w'_n)\right]dw'_1\cdots dw'_n \,,$$

and the path-ordering $\mathscr{P}$ forces an ordering of decreasing argument from left to right:

$$\mathscr{P}\left[a(w_1)a(w_2)\right] = \begin{cases} a(w_1)a(w_2) & w_1 > w_2 \\ a(w_2)a(w_1) & w_1 < w_2 \end{cases} .$$

Re-expressing $\tilde{\Psi}$ in terms of Ψ, we obtain

$$\Psi_j(w|\lambda) = \pi_j \left\{ e^{\phi(w)\, H} \mathscr{P} \exp \left[\lambda \int_0^w dw' \left(e^{-2\phi(w')} E + e^{2\phi(w')} F \right) \right] \right\} \Psi_j^0 \ ,$$

Now, for each representation π_j we can define a monodromy matrix:

$$\mathbf{M}_j(\lambda) = \pi_j \left\{ e^{2\pi i p\, H} \mathscr{P} \exp \left[\lambda \int_0^{2\pi} dw \left(e^{-2\phi(w)} E + e^{2\phi(w)} F \right) \right] \right\} \ , \tag{2.10}$$

and a corresponding L-matrix[6]

$$\mathbf{L}_j(\lambda) \doteq \pi_j \left[e^{-\pi i p\, H} \right] \mathbf{M}_j(\lambda) \ . \tag{2.11}$$

This last matrix can be shown to satisfy the r-matrix Poisson relation:

$$\left\{ \mathbf{L}_j(\lambda) \overset{\otimes}{,} \mathbf{L}_{j'}(\lambda') \right\}_P = \left[\mathbf{r}_{j\, j'}(\lambda/\lambda'),\, \mathbf{L}_j(\lambda) \otimes \mathbf{L}_{j'}(\lambda') \right] \ , \tag{2.12}$$

where the r-matrix is defined as

$$\mathbf{r}_{j\, j'}(\lambda) \doteq (\pi_j \otimes \pi_{j'}) \left[\mathbf{r}(\lambda) \right] \ , \quad \mathbf{r}(\lambda) \doteq \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}} \frac{H \otimes H}{2} + \frac{2}{\lambda - \lambda^1} \left(E \otimes F + F \otimes E \right) \ .$$

The r-matrix Poisson algebra tells us immediately that the quantities[7]

$$\mathbf{T}_j(\lambda) = tr \mathbf{M}_j(\lambda) \ ,$$

are in involution with respect with the Poisson brackets

$$\left\{ \mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda') \right\}_P = 0 \ ,$$

and are expected to generate the classical IMs in their asymptotic limit. Note that $\mathbf{T}_{\frac{1}{2}}(\lambda) = \mathbf{T}(\lambda)$.

We will not be showing the explicit proof of the relation (2.12), however we wish to close this section suggesting an approach to the computation which we think gives an intuitive interpretation of the form of $\mathbf{M}_j$. The core of this approach resides in the following expansion of the path-ordered exponential as a "continuum limit" of an ordered product:

$$\mathscr{P} \exp \left[\int_0^{2\pi} a(w') dw \right] = \lim_{N \to \infty} e^{a(w_N)\Delta w} e^{a(w_{N-1})\Delta w} \cdots e^{a(w_0)\Delta w} \ ,$$

where $w_j = j\Delta w$ and $\Delta w = \frac{2\pi}{N}$. This expression allows us to write $\mathbf{L}_j$ as (here too we omit π_j for clarity)

$$\mathbf{L}(\lambda) = e^{\pi i p\, H} \lim_{N \to \infty} \mathbf{l}(w_N|\lambda) \mathbf{l}(w_{N-1}|\lambda) \cdots \mathbf{l}(w_0|\lambda) \ ,$$

where the matrices $\mathbf{l}_j$ are

$$\mathbf{l}(w|\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n \Delta w^n}{n!} \left(e^{-2\phi(w)} E + e^{2\phi(w)} F \right)^n \ .$$

It is now sufficient to show that $\mathbf{l}$ satisfies the relation (2.12), for any value of w and λ. This is reminiscent of the approach to lattice models, in which we have a matrix $\mathbf{l}$ on each site and the full transfer matrix of the system is built as a trace of the product of these matrices for each site. In fact we can interpret the matrix $\mathbf{L}(\lambda)$ as the "continuum limit" of the monodromy matrix of a lattice model, where $e^{\pi i p H}$ plays the role of twist.

[6] In the literature this object is sometimes called Lax matrix.

[7] The operator denoted here with $\mathbf{T}$ are not to be confused with the densities of IM (2.1) introduced above.

2.3 The quantum monodromy matrix and the T-operators

Let us now concentrate on our goal: we are going to reproduce in the $c < -2$ CFTs[8] what has been sketched above for the classical KdV hierarchy. Namely we are going to address the problem of simultaneous diagonalisation of the local IMs (2.2-2.4) with a method that can be interpreted as a version of the Quantum Inverse Scattering (QIS) [17] for field theories. Just as it was suggested in the previous section, all the object we are going to introduce have a counterpart in lattice models and it is a good idea to keep in mind this parallelism. On the other hand these objects will be the quantised version of those introduced above for the classical KdV hierarchy and we are going to use the same symbols to denote them. Note that, from now on, we will consider the right chirality only.

In order to proceed to the construction of the objects $\mathbf{T}_j$, we first need the quantum version of Miura transformation: *the Feigin-Fuchs free field representation* [18]

$$-\beta^2 T(w) =: \left(\varphi'(w)\right)^2 : +(1-\beta^2)\varphi''(w) + \frac{\beta^2}{24} , \qquad \beta \doteq \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}} , \tag{2.13}$$

where $\varphi(w)$ is a free field

$$\varphi(w) = \mathrm{i}\, Q + \mathrm{i}\, P\, w + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{\mathrm{i}\, n\, w} ,$$

and the normal ordering $: \cdot :$ consist in placing the a_n oscillators in increasing n from left to right. The operators Q, P and $\{a_n\}_{n\neq 0}$ generate an Heisenberg algebra:

$$[Q,P] = \frac{\mathrm{i}}{2}\beta^2 , \qquad [a_n, a_m] = \frac{n}{2}\beta^2 \delta_{n+m,0} .$$

It is easy to see that this transformation becomes exactly (2.9) as $c \to -\infty$. With this expression for $T(w)$ we are able to give a description of the Hilbert space $\mathscr{H}_{ch}$ in terms of Fock spaces $\mathscr{F}_p$, defined as highest-weight modules over the Heisenberg algebra; the highest-weight vector $|\, p\,\rangle \in \mathscr{F}_p$ obeys to the following relations

$$P\,|\, p\,\rangle = p\,|\, p\,\rangle \ , \qquad a_n\,|\, p\,\rangle = 0 \ , \ \forall n > 0 .$$

The Fock space thus defined is isomorphic to the Verma module $\mathcal{V}_h$, where

$$h = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24} ,$$

and we can describe the Hilbert space as

$$\mathscr{H}_{ch} = \bigoplus_a \mathscr{F}_a , \qquad \mathscr{F}_a \equiv \mathscr{F}_{p_a} ,$$

where the direct sum runs over the values of p corresponding to the allowed Virasoro highest-weights. These Fock spaces are naturally graded under the action of L_0:

$$\mathscr{F}_p = \bigoplus_{\ell=0}^{\infty} \mathscr{F}_p^{(\ell)} , \qquad L_0 \mathscr{F}_p^{(\ell)} = (h+\ell)\mathscr{F}_p^{(\ell)} .$$

With some simple algebraic manipulation we can express the Virasoro generators $\{L_n\}$ in terms of the Heisenberg algebra as

$$\begin{aligned} \beta^2 L_n &= \beta^2 \frac{c-1}{24} + 2\sum_{j\neq 0,n} a_j a_{n-j} + a_n\left(2P - n(1-\beta^2)\right) , \\ \beta^2 L_0 &= \beta^2 \frac{c-1}{24} + 2\sum_{j=1}^{\infty} a_{-j} a_j + P^2 . \end{aligned}$$

[8] The restriction to this domain will be clearer later.

Since in theory we know how to express the local IMs $\{\mathbf{I}_{2k-1}\}$ in terms of the local densities $T_{2k}(w)$, the formers can be re-expressed in terms of polynomials in the free field $\varphi(w)$ and its derivatives:

$$\mathbf{I}_{2k-1} = (-1)^k \beta^{-2k} \int_0^{2\pi} \frac{dw}{2\pi} \left[:(\varphi'(w))^{2k}: + \underbrace{\cdots\cdots\cdots\cdots}_{\text{Higher derivatives of } \varphi(w)} \right] .$$

As it is evident from their definition (remember the spin assignment request), each term in a local IM, as complicated as it might be, is nevertheless a product of operators L_{n_i}, where the sum of indices vanishes: $\sum_i n_i = 0$. As a consequence $[L_0, \mathbf{I}_{2k-1}] = 0$ and the local IMs act invariantly on the level subspaces $\mathscr{F}_p^{(\ell)}$. The full diagonalisation of the integrals of motion is thus reduced to their diagonalisation on each level subspace, which requires a finite number of algebraic manipulations; these, however, become rapidly extremely involved and so far the result is known only for some simple cases, e.g. for the vacuum $|p\rangle$

$$\begin{aligned} I_1^{(\text{vac})}(h,c) &= h - \frac{c}{24} , \\ I_3^{(\text{vac})}(h,c) &= h^2 - \frac{c+2}{12} h + c \frac{5c+22}{4 \times 6!} , \\ I_5^{(\text{vac})}(h,c) &= h^3 - \frac{c+4}{8} h^2 + 5(c+2) \frac{3c+20}{4 \times 6!} h - 5c(3c+14) \frac{7c+68}{4 \times 9!} , \\ &\cdots \end{aligned}$$

where $\mathbf{I}_{2k-1} \, | \, p \, \rangle = I_{2k-1}^{(\text{vac})} \, | \, p \, \rangle$.

In the setting provided by the Fock description of the Hilbert space, we can easily follow the footprints of section 2.2 and define quantum counterparts of the monodromy matrices (2.10) and of the L-matrices (2.11). In order to do so, we consider the quantum enveloping algebra $\mathcal{U}_q\left(sl(2)\right)$ [19, 20] generated by the elements E, F and H:

$$[H, E] = 2E , \quad [H, F] = -2F , \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}} ,$$

with

$$q \doteq e^{\mathrm{i}\pi\beta^2} ,$$

and let π_j denote the $(2j+1)$-dimensional representation of this algebra. The "quantum monodromy matrices" are then defined as the following operator-valued matrices[9]

$$\mathbf{M}_j(\lambda) \doteq \pi_j \left\{ e^{2\pi i P\, H} \mathscr{P} \exp \left[\lambda \int_0^{2\pi} dw \left(V_-(w) q^{\frac{H}{2}} E + V_+(w) q^{-\frac{H}{2}} F \right) \right] \right\} , \tag{2.14}$$

where

$$V_\pm(w) \doteq :e^{\pm 2\varphi(w)}: = \exp \left[\pm 2 \sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{\mathrm{i}nw} \right] e^{\pm 2\mathrm{i}(Q+Pw)} \exp \left[\mp 2 \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-\mathrm{i}nw} \right] ,$$

are called vertex operators; they have conformal dimension β^2 and act on the Fock spaces by shifting the highest weight:

$$V_\pm(w) \ : \quad \mathscr{F}_p \longrightarrow \mathscr{F}_{p \pm \beta^2} .$$

The reason why in the path ordered exponent the combinations $q^{\frac{H}{2}} E$ and $q^{-\frac{H}{2}} F$ appear is related to the fact that the correct construction of $\mathbf{M}_j$ should start from the quantum affine enveloping algebra $\mathcal{U}_q\left(\widehat{sl(2)}\right)$; we will return to this point in section 2.5.

The L-operators are defined in the same way as in the classical case:

$$\mathbf{L}_j(\lambda) \doteq \pi_j \left[e^{-\pi i P\, H} \right] \mathbf{M}_j(\lambda) . \tag{2.15}$$

[9] Note that these objects can be informally interpreted as the monodromy matrices of the solution Ψ_j to the operator-matrix equation $\pi_j \left[\mathscr{L}(w|\lambda) \right] \Psi_j(w|\lambda) = 0$, where $\mathscr{L}(w|\lambda) = \partial - \varphi'(w)\, H - \lambda \left(q^{\frac{H}{2}} E + q^{-\frac{H}{2}} F \right)$. Here one has to take care of normal ordering when defining the corresponding of $\tilde{\Psi} =: e^{-\varphi} : \Psi$. This gives rise to the presence of vertex operators in the path-ordered integral.

Both the quantum monodromy matrices and the L-operators are $(2j+1)\times(2j+1)$ matrices whose elements are operators acting on the space

$$\hat{\mathscr{F}}_p \doteq \bigotimes_{n=-\infty}^{\infty} \mathscr{F}_{p+n\beta^2} \ .$$

They have to be understood as power series in λ:

$$\mathbf{L}_j(\lambda) = \pi_j \left[e^{\pi i P\, H} \sum_{k=0}^{\infty} \lambda^k \int_{\substack{0 \\ w_1 \geq \cdots \geq w_k}}^{2\pi} dw_1 \cdots dw_k \mathcal{K}(w_1) \cdots \mathcal{K}(w_k) \right] ,$$

where we introduced

$$\mathcal{K}(w) \doteq V_-(w) q^{\frac{H}{2}} E + V_+(w) q^{-\frac{H}{2}} F \ . \tag{2.16}$$

These series converge for any λ if[10] $-\infty < c < -2$; outside this region the definition of $\mathbf{M}_j$ and $\mathbf{L}_j$ necessitate a proper regularisation. In these notes we will limit ourselves to the cases $c < -2$.

The operators $\mathbf{L}_j$ are tailored in such a way that the following RLL relation is satisfied

$$\mathbf{R}_{j\,j'}(\lambda/\lambda') \left(\mathbf{L}_j(\lambda) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \mathbf{L}_{j'}(\lambda')\right) = \left(\mathbb{I} \otimes \mathbf{L}_{j'}(\lambda')\right) \left(\mathbf{L}_j(\lambda) \otimes \mathbb{I}\right) \mathbf{R}_{j\,j'}(\lambda/\lambda') \ , \tag{2.17}$$

where $\mathbf{R}_{j\,j'}(\lambda)$ is the trigonometric R-matrix of $\mathcal{U}_q\left(sl(2)\right)$, acting on $\pi_j \otimes \pi_{j'}$. The fundamental-fundamental case $j = j' = \frac{1}{2}$ reads as follows[11]

$$\mathbf{R}_{\frac{1}{2}\frac{1}{2}}(\lambda) = \begin{pmatrix} \frac{\lambda}{q} - \frac{q}{\lambda} & & & \\ & \lambda - \lambda^{-1} & q^{-1} - q & \\ & q^{-1} - q & \lambda - \lambda^{-1} & \\ & & & \frac{\lambda}{q} - \frac{q}{\lambda} \end{pmatrix} . \tag{2.18}$$

In order to check the validity of the RLL relation it is possible to adopt a brute force method, that is discretise the $\mathscr{P}$ exponential and compute the two sides of the relation, or interpret $\mathbf{L}_j$ and $\mathbf{R}_{j\,j'}$ as particular realisations of universal objects of the algebra $\mathcal{U}_q\left(\widehat{sl(2)}\right)$; this last approach is sketched in section 2.5.

We can finally define the "quantum transfer matrices" as traces of the quantum monodromy matrices (2.14):

$$\mathbf{T}_j(\lambda) \doteq \mathrm{tr}_{\pi_j}\left(\mathbf{M}_j(\lambda)\right) \ . \tag{2.19}$$

As a direct consequence of the RLL relation, these matrices form a commuting family:

$$[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\lambda')] = 0 \ ,$$

moreover they commute with the operator P and, as such, act invariantly[12] on each $\mathscr{F}_p$. Finally through some tedious computation [4] it is possible to show that, with the definition (2.19), the quantum transfer matrices commute with all the local IMs[13]:

$$[\mathbf{T}_j(\lambda), \mathbf{I}_{2k-1}] = 0 \ ,$$

which means that the level subspaces $\mathscr{F}_p^{(\ell)}$ are the eigenspaces of $\mathbf{T}_j(\lambda)$.

Before explicitly presenting the simple case of $\pi_j = \pi_{\frac{1}{2}}$, we wish to underline the connection with lattice models. Just as in the classical case, we can express the $\mathbf{L}$-operators as a continuum limit of a product:

$$\mathbf{L}(\lambda) = e^{\pi i P H} \lim_{N\to\infty} \mathbf{l}(w_N|\lambda)\mathbf{l}(w_{N-1}|\lambda)\cdots\mathbf{l}(w_0|\lambda) \ .$$

[10] This is most easily inferred from the fact that the vertex operators $V_\pm$ have conformal dimension β^2 and, thus, $V_+(w)V_-(w') \sim (w-w')^{-2\beta^2}\left(1 + O(w-w')\right)$. So, for the integrals to converge we must impose $\beta^2 < \frac{1}{2}$, which is equivalent to $c < -2$.

[11] Note that this is the same exact matrix as for the 6-vertex model [21]!

[12] An informal way to see this is to notice that the operators $\mathcal{K}(w)$ are traceless; moreover the only terms in $\mathbf{T}_j$ having non-vanishing trace are those containing an equal number of operators E and F. Thus, only products of elements of the type $V_-(w)V_+(w')$ appear and these act on Fock spaces as $\mathscr{F}_p \to \mathscr{F}_{p+\beta^2} \to \mathscr{F}_p$.

[13] Note that the proof is limited to some low order IM; a full proof of the commutativity is still lacking.

Here the "local" operators $\mathbf{l}$ are expressed as

$$\mathbf{l}(w|\lambda) = \exp\left[\lambda \mathcal{K}(w)\Delta w\right] \sim 1 + \lambda \mathcal{K}(w)\Delta w \ .$$

The form of $\mathcal{K}(w)$ is exactly that which we would expect from a lattice model:

$$\mathcal{K}(w) = \sum_{j=\pm} V_j(w)\omega_j \ ,$$

where $\omega_\pm$ are generators of the $\mathcal{U}_q\left(\widehat{sl(2)}\right)$ algebra in matrix realisation (see section 2.5 for more details), and $V_\pm$ are a vertex operator realisation of the same algebra. In fact the operators

$$V_0(w) = \sqrt{2}\partial_w \varphi(w) \ , \qquad V_\pm(w) =: e^{\pm 2\varphi(w)} \ ,$$

satisfy the $\widehat{sl(2)}$ subalgebra at level 1 [9]. So we can interpret $\mathbf{l}$ as being the tensor product of two operators acting on two different spaces: one, corresponding to the matrices ω_j, is the auxiliary space; the other, corresponding to the vertex operators V_j, is the quantum space. A pictorial representation is given in Figure 2.2. We wish to stress that this connection is by no means mathematically precise, but rather an intuitive interpretation of the physical meaning of the operators introduced above.

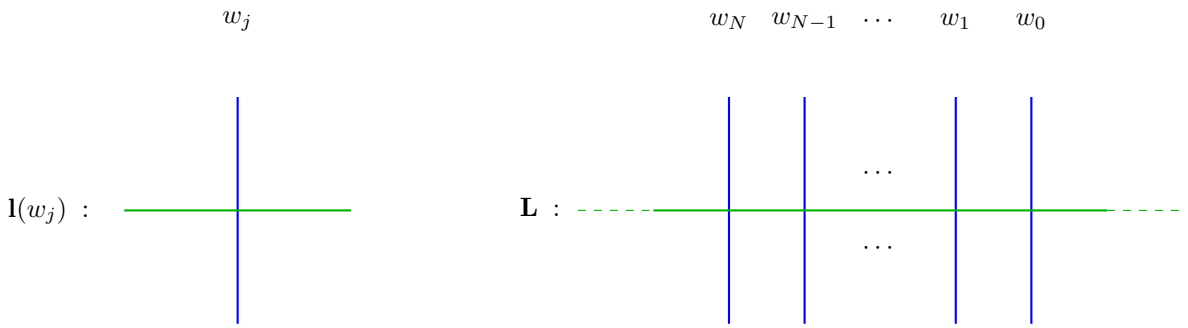


Fig. 2.2: Graphical representation of operators $\mathbf{l}(w_j)$ and $\mathbf{L}$; the horizontal green line represents the auxiliary space, while the vertical blue ones correspond to the quantum spaces.

The basic representation All that has been said until now is rather general and abstract. In order to make things more concrete, let us concentrate on the simplest amongst the quantum transfer matrices, namely $\mathbf{T}(\lambda) \equiv \mathbf{T}_{\frac{1}{2}}(\lambda)$. The 2-dimensional representation of $\mathcal{U}_q\left(sl(2)\right)$ can be chosen as

$$\pi_{\frac{1}{2}}[H] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad \pi_{\frac{1}{2}}[E] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \pi_{\frac{1}{2}}[F] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ,$$

and the operator $\mathbf{T}(\lambda)$ can thus be written as a power series in λ^2 (due to the tracelessness of the matrices E and F):

$$\mathbf{T}(\lambda) = 2\cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} \mathbf{G}_{2n} \ , \tag{2.20}$$

where

$$\begin{aligned} \mathbf{G}_{2n} \ \doteq \ q^n \int\limits_{\substack{0 \\ w_1 \geq \cdots \geq w_{2n}}}^{2\pi} dw_1 \cdots dw_{2n} \ e^{2\pi \mathrm{i} P} V_-(w_1)V_+(w_2)\cdots V_-(w_{2n-1})V_+(w_{2n}) + \\ + e^{-2\pi \mathrm{i} P} V_+(w_1)V_-(w_2)\cdots V_+(w_{2n-1})V_-(w_{2n}) \ , \end{aligned} \tag{2.21}$$

are operators commuting amongst themselves and with the local IMs

$$[\mathbf{G}_{2n}, \mathbf{G}_{2m}] = 0 \,, \qquad [\mathbf{G}_{2n}, \mathbf{I}_{2k-1}] = 0 \,,$$

and, for this reason, are called *non-local integrals of motion*. Just as their local counterpart, they act invariantly on each level subspace $\mathscr{F}_p$ and, in particular, the highest weight vector $|\, p \,\rangle$ is one of their common eigenstates:

$$\mathbf{G}_{2n} \,|\, p \,\rangle = G_{2n}^{(\text{vac})}(p) \,|\, p \,\rangle \ .$$

The vacuum eigenvalues can be calculated rather straightforwardly from the definition (2.21):

$$G_{2n}^{(\text{vac})}(p) = \int\limits_{\substack{0 \\ w_1 \geq \tilde{w}_1 \geq \cdots \geq w_n \geq \tilde{w}_n}}^{2\pi} dw_1 d\tilde{w}_1 \cdots dw_n d\tilde{w}_n \prod_{j>i\geq 1}^{n} \left[4 \sin\left(\frac{w_i - w_j}{2}\right) \sin\left(\frac{\tilde{w}_i - \tilde{w}_j}{2}\right) \right]^{2\beta^2} \times$$
$$\times \prod_{i,j=1}^{n} \left[2 \sin\left(\frac{w_i - \tilde{w}_j}{2}\right) \right]^{-2\beta^2} 2 \cos\left[2p \left(\pi + \sum_{j=1}^{n} (\tilde{w}_j - w_j) \right) \right] \,,$$

and when $n = 1$, this expression greatly simplifies to

$$G_2^{(\text{vac})}(p) = \int_0^{2\pi} dw \int_0^{w} d\tilde{w} \frac{2\cos\left(2\pi p + 2p(\tilde{w} - w)\right)}{\left[2\sin\left(\frac{w-\tilde{w}}{2}\right)\right]^{2\beta^2}} = \frac{4\pi^2 \Gamma\left(1-\beta^2\right)}{\Gamma\left(1-2p-\beta^2\right)\Gamma(1+2p-\beta^2)} \,.$$

In order to obtain these results one has to use the following property of vertex operators:

$$\langle p |\, V_\epsilon(w) V_{\tilde{\epsilon}}(\tilde{w}) \,| p \rangle = e^{-2\mathrm{i}p(\tilde{\epsilon}w + \epsilon\tilde{w})} \left[2 \sin\left(\frac{w - \tilde{w}}{2}\right) \right]^{-2\epsilon\tilde{\epsilon}\beta^2} \,, \qquad \epsilon, \tilde{\epsilon} = \pm 1 \,,$$

and the Wick theorem.

All the operators $\mathbf{T}_j(\lambda)$ are entire functions of λ^2, possessing an essential singularity at infinity due to the accumulation of zeroes on the negative λ^2-axis. This can be shown by comparison with a result obtained in [22], where a series similar to (2.20) was analysed and shown to converge in the whole complex plane, defining an entire function with an essential singularity at infinity. As it turns out, the coefficients of this series are larger in absolute value than (2.21), meaning that $\mathbf{T}$ is entire as well. Finally the entirety of the operators $\mathbf{T}_j$ directly descends from this result, thanks to the T-system we are going to present just below (2.25).

We are interested in obtaining an asymptotic series expansion of $\mathbf{T}(\lambda)$ since, recalling equation (2.8), we expect the integrals of motion to appear there as coefficients. In fact, as pointed out just above, it is possible to examine the discretised version of $\mathbf{M}_{\frac{1}{2}}(\lambda)$. Then, by means of standard Algebraic Bethe Ansatz (ABA), and, subsequently, taking the continuum limit back to $\mathbf{M}_{\frac{1}{2}}(\lambda)$ one obtains the following expression for the quantum transfer matrix:

$$\mathbf{T}(\lambda) = \mathbf{\Lambda}(q\lambda) + \mathbf{\Lambda}^{-1}(q^{-1}\lambda) \,,$$

where

$$\log \mathbf{\Lambda}(q\lambda) \underset{\substack{|\lambda|\to\infty \\ |\arg(\lambda)|<\pi}}{\sim} m\lambda^{1+\xi} - \sum_{k=1}^{\infty} C_k \mathbf{I}_{2k-1} \lambda^{(1+\xi)(1-2k)} \,, \qquad \xi \doteq \frac{\beta^2}{1-\beta^2} \,.$$

The constants in the asymptotic expansion are

$$m = 2\sqrt{\pi} \frac{\Gamma\left(\frac{1}{2} - \frac{\xi}{2}\right)}{\Gamma\left(1 - \frac{\xi}{2}\right)} \left[\Gamma\left(\frac{1}{1+\xi}\right) \right]^{1+\xi} \,, \tag{2.22}$$

$$C_k = \frac{1+\xi}{k!} \left(\frac{\pi\xi}{1+\xi} \right)^k \left(\frac{2}{C_0} \frac{\Gamma\left(\frac{1}{2} - \frac{\xi}{2}\right)}{\Gamma\left(1 - \frac{\xi}{2}\right)} \right)^{2k-1} \frac{\Gamma\left[(1+\xi)(k - \frac{1}{2})\right]}{\Gamma\left[1 + (k - \frac{1}{2})\xi\right]} \,. \tag{2.23}$$

Notice how in this asymptotic expansion, λ appears with fractional powers $1+\xi$; this might seem surprising, until one remembers that the vertex operators $V_\pm(w)$ carry a conformal dimension $h_V = \beta^2 = \frac{\xi}{1+\xi}$. We can thus think to the spectral parameter λ as carrying an anomalous dimension $[\lambda] = [\text{length}]^{-\frac{1}{1+\xi}}$. The explicit computations which yield these results are rather lengthy and we will not present them here. Nonetheless we encourage the interested reader to delve into them, starting from the above cited paper [22].

2.4 T-system, Y-system and Thermodynamic Bethe Ansatz equations

Let us return to the analysis of the quantum monodromy matrices $\mathbf{T}_j(\lambda)$ associated highest dimensional representations of $\mathcal{U}_q(sl(2))$. It is easily deduced from their definition that them too are power series in λ^2:

$$\mathbf{T}_j(\lambda) = \frac{\sin(2(2j+1)\pi P)}{\sin(2\pi P)} + \sum_{n=1}^{\infty} \lambda^{2n} \mathbf{G}_{2n}^{(j)} \ .$$

The surprising fact about these expansions is that they are deeply interrelated; in fact the non-local IMs $\mathbf{G}_{2n}^{(j)}$ with $j > \frac{1}{2}$ can all be written as polynomials in $\mathbf{G}_{2n}^{(\frac{1}{2})} \equiv \mathbf{G}_{2n}$, e.g.

$$\begin{aligned} \mathbf{G}_2^{(j)} &= A_j(2\pi P, \pi\beta^2)\mathbf{G}_2 \ , \\ \mathbf{G}_4^{(j)} &= A_j(2\pi P, 2\pi\beta^2)\mathbf{G}_4 + B_j(2\pi P, \pi\beta^2)\mathbf{G}_2 \ , \\ &\cdots \quad , \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} A_j(a,b) &\doteq \frac{1}{4\sin a \sin b}\left[\frac{\sin[(2j+1)(a-b)]}{\sin(a-b)} - \frac{\sin[(2j+1)(a+b)]}{\sin(a+b)}\right] \ , \\ B_j(a,b) &\doteq \frac{1}{16\sin a\sin b\sin 2b}\left[\frac{\sin[(2j+1)(a-2b)]}{\sin(a-b)\sin(a-2b)} + \frac{\sin[(2j+1)(a+2b)]}{\sin(a+b)\sin(a+2b)} + \right. \\ &\qquad \left. -2\cos b\frac{\sin[(2j+1)a]}{\sin(a-b)\sin(a+b)}\right] \ . \end{aligned}$$

These polynomial relations suggest that there might exist algebraic relation between quantum transfer matrices belonging to different representations π_j. This is indeed the case, as the operators $\mathbf{T}_j$ satisfy the following system of finite-difference functional equations

$$\mathbf{T}_j(q^{\frac{1}{2}}\lambda)\mathbf{T}_j(q^{-\frac{1}{2}}\lambda) = 1 + \mathbf{T}_{j+\frac{1}{2}}(\lambda)\mathbf{T}_{j-\frac{1}{2}}(\lambda) \ , \tag{2.25}$$

known as *T-system* or *Hirota bilinear equations* [23, 24]. This system of equations is a direct consequence of the RLL relation (2.17) and can be obtained by using a procedure called R-matrix fusion[14], well known in lattice theory [19]. It is worth noticing that equations (2.17), (2.25) and the R-matrix (2.18) are essentially the same as the corresponding ones in the integrable XXZ model [21]. This, clearly, is not just a coincidence as the underlying algebraic structure of the latter is the same as that of the CFTs we are studying here; this structure knows nothing about the discrete or continuous nature of the system and is thus expected that the equations arising from purely algebraic considerations (such as the RLL relation above or, as we will see, the TQ equation) have the same structure, no matter what is the model under study. The information on the different nature of the models will be contained then in the analytical properties of the objects involved in these relations. These considerations will be precious later, when we will extend this setting to massive theories.

For generic values of the central charge c, we have an infinite hierarchy of quantum transfer matrices which, thanks to the system (2.25), can all be expressed in terms of the fundamental one $\mathbf{T}(\lambda)$:

$$\begin{aligned} \mathbf{T}_1(\lambda) &= \mathbf{T}(q^{\frac{1}{2}}\lambda)\mathbf{T}(q^{-\frac{1}{2}}\lambda) - 1 \ , \\ \mathbf{T}_{\frac{3}{2}}(\lambda) &= \mathbf{T}(q\lambda)\mathbf{T}(\lambda)\mathbf{T}(q^{-1}\lambda) - \mathbf{T}(q^{\frac{1}{2}}\lambda) - \mathbf{T}(q^{-\frac{1}{2}}\lambda) \ , \\ &\cdots \quad . \end{aligned}$$

[14] In fact, while the expressions (2.24) can be obtained by brute force computation, we prefer to consider it as a consequence of (2.25).

With some algebraic effort, we can also recast the T-system in the following form

$$\mathbf{T}(\lambda)\mathbf{T}_j(q^{\frac{2j+1}{2}}\lambda) = \mathbf{T}_{j-\frac{1}{2}}(q^{\frac{2j+2}{2}}\lambda) + \mathbf{T}_{j+\frac{1}{2}}(q^{\frac{2j}{2}}\lambda) \ . \tag{2.26}$$

Let us now introduce the Y-operators as follows [25]:

$$\mathbf{Y}_j(\theta) \doteq \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda) \ , \qquad \lambda^{1+\xi} = e^{\theta} \ ,$$

with the convention $\mathbf{T}_0 = 1$ and $\mathbf{T}_{-\frac{1}{2}} = 0$; then it is easily showed that they satisfy the Y-system equations

$$\mathbf{Y}_j^+\mathbf{Y}_j^- = \left(1+\mathbf{Y}_{j+\frac{1}{2}}\right)\left(1+\mathbf{Y}_{j-\frac{1}{2}}\right) \ , \tag{2.27}$$

where we have introduced the short-hand notation for shifts: $\mathbf{Y}^{\pm} \doteq \mathbf{Y}(\theta \pm \mathrm{i}\pi\frac{\xi}{2})$. This last infinite system of finite difference equation can be further recast in an infinite set of non-linear integral equations, known as *Thermodynamic Bethe Ansatz equations* whose general form is the following

$$\epsilon_j^{(\ell)}(\theta) = \mathcal{Z}^{(\ell)}(\theta) - \sum_k \int_{-\infty}^{\infty} d\theta' \, \varphi_{j\,k}\,(\theta-\theta') \log\left[1+e^{-\epsilon_k^{(\ell)}(\theta')}\right] \ , \tag{2.28}$$

where the *pseudo-energies* $\epsilon_j^{(\ell)}$ are the logarithms of the Y-operators eigenvalues:

$$\epsilon_j^{(\ell)}(\theta) \doteq \log\left[Y_j^{(\ell)}(\theta)\right] \ , \qquad \mathbf{Y}_j(\theta)\,|\ell\rangle = Y_j^{(\ell)}(\theta)\,|\ell\rangle \ ,$$

and ℓ labels the eigenstate under consideration. The function $\mathcal{Z}^{(\ell)}(\theta)$ is called *driving term* and depends on the particular eigenstate, while the kernel $\varphi_{j\,k}(\theta)$ only depends on the algebraic structure of the Y-system. The procedure to go from (2.27) to (2.28) is intuitively simple, however one has to take great care to the analytic properties of the functions involved. More specifically one has to know the asymptotic behaviour of the Y-functions, which will be encoded into the function $\mathcal{Z}^{(\ell)}$. Moreover the presence of poles and zeroes in the functions $Y_j^{(\ell)}$ might create a great deal of problems. All these questions are addressed in the [26] and we recommend interested readers to refer to that review.

Truncation and the minimal models $\mathcal{M}_{2,2n+1}$ TBA The relations we derived just above, the T-system, the Y-system and the TBA equations, although very simple-looking and fancy, still consists of an infinity of equations for an infinite set of functions; for this reason, dealing with them is, to use an euphemism, complicated. However there are situations in which the number of equations and functions involved reduce to a finite number; this fenomenon is called *truncation*. The parallel we traced above with the lattice model helps us identify these cases: it is known that, for some particular values of the parameters, the XXZ system can be reduced to the $RSOS$ model [27] and the T-system collapses to a finite set of equations for a finite number of functions [28, 29]. This phenomenon of truncation in XXZ can be traced back to a purely algebraic fact: when q is a N-th root of unity the $(N+1)$-dimensional representation $\pi_{\frac{N}{2}}$ of $\mathcal{U}_q\,(sl(2))$ becomes reducible, while all the representations with $\frac{1}{2} \leq j < \frac{N}{2}$ remain irreducible. In particular

$$\pi_{\frac{N}{2}} = \pi_{\frac{N}{2}-1} \oplus \varrho_N^+ \oplus \varrho_N^- \ ,$$

where $\varrho_N^{\pm}$ are two particular one-dimensional representations such that

$$\varrho_N^{\pm}\,[E] = \varrho_N^{\pm}\,[F] = 0 \ , \qquad \varrho_N^{\pm}\,[H] = \pm N \ .$$

Being purely algebraic and, as such, blind to the particular theory overlying the algebra structure, we expect the phenomenon of truncation to happen for CFTs as well. Indeed, considering the decomposition above, and applying it to the abstract definition 2.14 we immediately obtain that

$$\mathbf{T}_{\frac{N}{2}}(\lambda) = 2\cos\left(2\pi N P\right) + \mathbf{T}_{\frac{N}{2}-1}(\lambda) \ ,$$

which makes (2.25) a closed set of equations for the operators $\{\mathbf{T}_j\}_{j=0}^{\frac{N}{2}-\frac{1}{2}}$.

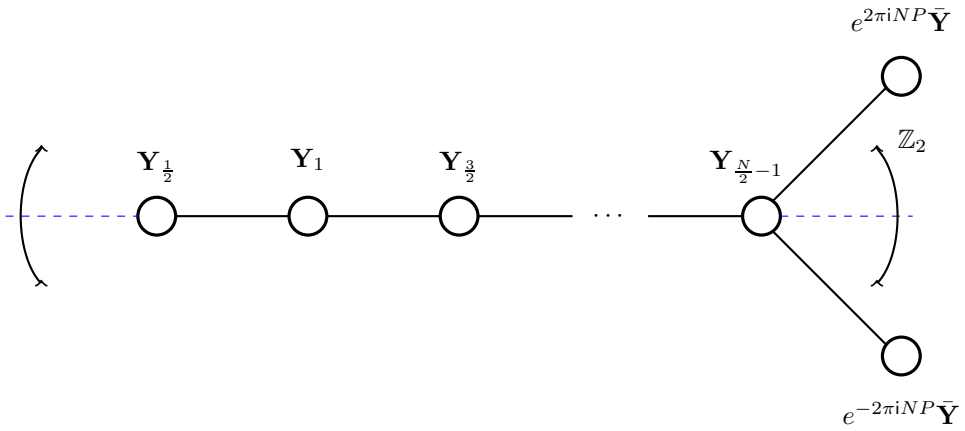
In this case too is convenient to introduce the Y-operators, with a slight modification with respect to the general case, due to the finiteness of the system:

$$\begin{aligned}\mathbf{Y}_j(\theta) &\doteq \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda)\,, \quad j=\frac{1}{2},1,\dots,\frac{N}{2}-1\,,\\ \mathbf{Y}_0(\theta) &\doteq 0\,,\\ \overline{\mathbf{Y}}(\theta) &\doteq \mathbf{T}_{\frac{N}{2}-1}\,.\end{aligned}$$

The Y-system is then immediately seen to be as follows

$$\begin{aligned}\mathbf{Y}_j^+\mathbf{Y}_j^- &= \left(1+\mathbf{Y}_{j-\frac{1}{2}}\right)\left(1+\mathbf{Y}_{j+\frac{1}{2}}\right)\,, \qquad j=\frac{1}{2},1,\dots,\frac{N}{2}-\frac{3}{2}\,,\\ \mathbf{Y}^+_{\frac{N}{2}-1}\mathbf{Y}^-_{\frac{N}{2}-1} &= \left(1+\mathbf{Y}_{\frac{N}{2}-\frac{3}{2}}\right)\left(1+e^{2\pi i NP}\overline{\mathbf{Y}}\right)\left(1+e^{-2\pi i NP}\overline{\mathbf{Y}}\right)\,,\\ \overline{\mathbf{Y}}^+\overline{\mathbf{Y}}^- &= \left(1+\mathbf{Y}_{\frac{N}{2}-1}\right)\,.\end{aligned} \tag{2.29}$$

This Y-system is called of type D_N [30] as it can be nicely encoded in the Dynkin diagram of said type. Indeed let us associate each Y-operator with a node of a graph and draw lines between these whenever the corresponding Ys appear in the same equation. What we obtain is the diagram shown in the picture below: a Dynkin diagram of type D_N.



Note that for q to be a root of unity, we must require β^2 to be a rational number, say $\beta^2=\frac{m'}{m}$ and, by virtue of (2.13), the central charge becomes

$$c=13-6\left(\beta^{-2}+\beta^2\right)=1-6\frac{(m-m')^2}{m\,m'}\,,$$

identifying our CFT as the minimal model $\mathcal{M}_{m,m'}$. Thanks to this identification we now realize that the truncation of the T-system is a clear reflection of the finite number of primary fields of these CFTs.

We can get a further simplification of (2.29) by considering the particular minimal models $\mathcal{M}_{2,2n+3}$

$$c=1-3\frac{(2n+1)^2}{2n+3}\,,\quad \beta^2=\frac{2}{2n+3}\,,\quad p_k=\frac{2k-2n-3}{2(2n+3)}\,,\quad k=1,\dots,n+1\,,$$

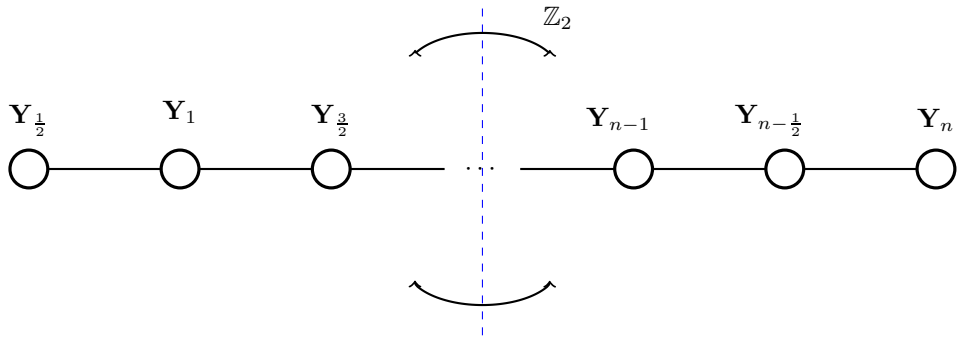
for which the Kač table is composed of a single line of $2(n+1)$ boxes, symmetric along the middle. This symmetry reflects itself in the T-system and the following relation is valid

$$\mathbf{T}_{n+\frac{1}{2}-1}(\lambda)=\mathbf{T}_j(\lambda)\,,\quad j=0,\frac{1}{2},\dots,n+\frac{1}{2}\implies \mathbf{T}_{n+\frac{1}{2}}(\lambda)=1\,,$$

thanks to this which, the Y-system simplifies to

$$\begin{aligned} \mathbf{Y}_j^+ \mathbf{Y}_j^- &= \left(1+\mathbf{Y}_{j-\frac{1}{2}}\right)\left(1+\mathbf{Y}_{j+\frac{1}{2}}\right) , \qquad j=\frac{1}{2},1,\dots,n \\ \mathbf{Y}_{n+\frac{1}{2}-j} &= \mathbf{Y}_j , \end{aligned}$$

which corresponds to the Dynkin diagram of A_{2n} type [30, 31], depicted in the following figure.



Let us now focus on the particular eigenvalues $Y_j^{\text{gr.st}}(\theta)$ corresponding to the ground state $|\,p_{n+1}\,\rangle$ (this is the state with lowest L_0 eigenvalue). We know that the functions $T_j^{\text{gr.st.}}(\lambda)$ are entire functions of λ^2, with asymptotic behaviour

$$T_j^{\text{gr.st.}}(\lambda) \underset{\lambda\to\infty}{\sim} m_j\,\lambda^{1+\xi} , \qquad m_j = \frac{2m}{\pi}\cot\left(\frac{\pi}{2}\xi\right)\sin\left(\pi j\xi\right) ,$$

with m given in (2.22). This information is sufficient to pass from the Y-system to the TBA equations:

$$\varepsilon_j(\theta) = \pi m_j e^{\theta} - \sum_{j'}\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\varphi_{j\,j'}(\theta-\theta')\log\left[1+e^{-\varepsilon_{j'}(\theta')}\right] , \tag{2.30}$$

where we introduced the *pseudo-energies* ε_j as

$$Y_j^{\text{gr.st.}}(\theta) = e^{\epsilon_j(\theta)} .$$

The kernel $\varphi_{j\,j'}(\theta)$ is defined from the equation

$$(1-\varphi)^{-1} = 1 - s\,\hat{I} \,, \Rightarrow\ \varphi_{j\,k}(\theta) = -s(\theta)\,\hat{I}_{j\,k} + \sum_i \hat{I}_{j\,i}\int_{-\infty}^{\infty} s(\theta'-\theta)\varphi_{i\,k}(\theta')d\theta' ,$$

with $\hat{I}_{j\,k}$ being the incidence matrix of A_{2n} and $s(\theta)=\frac{1}{\xi\cosh(\theta/\xi)}$ the inverse of the shift operator: $s^{-1}\ :\ f\to f^+ + f^-$. Taking the Fourier transform of the above relation, with some effort, is possible to show that the kernel φ can be expressed as the logarithmic derivative of the "massless S-matrix" $S_{j\,j'}(\theta)$ [31]

$$\varphi_{j\,j'}(\theta) \doteq -\mathrm{i}\partial_\theta \log\left(S_{j\,j'}(\theta)\right) ,$$

whose explicit form is known:

$$S_{j\,j'}(\theta) = F_{j+j'}(\theta)F_{|j-j'|}(\theta)\prod_{k=1}^{2\min(j,j')-1} F^2_{|j-j'|+k}(\theta) , \quad F_j(\theta) \doteq \frac{\sinh\theta+\mathrm{i}\sin\left(\pi j\xi\right)}{\sinh\theta-\mathrm{i}\sin\left(\pi j\xi\right)} .$$

Notice that this matrix appears directly from the algebraic properties of the truncated Y-system. It is a consequence of the internal consistency of this setting that $S_{j\,j'}$ happens to be exactly the two-particle element of the factorisable scattering matrix proposed in [32] for the S-matrix[15]description of minimal models of the type $\mathcal{M}_{2,2n+1}$. This little "miracle" gives us a strong confirmation of the correctness of the BLZ approach.

2.5 Baxter Q-operators

The construction of the Q-operators follows very closely that of the T-operators presented above. Like these last they are defined as traces of some particular monodromy matrix built out of vertex operators and the generators of some algebra. The difference between the two stands exactly in the choice of the algebra. For the construction of Q-operators it turns out that we need the quantum oscillator algebra osc_q generated by $\{\mathcal{H}, \mathcal{E}_+, \mathcal{E}_-\}$ with commutation relations

$$[\mathcal{H}, \mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm \ , \qquad q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}} \ .$$

The appearance of this algebra might seems strange as, at first sight, it does not seem to be related to the $sl(2)$ algebraic structure we have been using to construct everything else. Truth is, osc_q and $sl(2)$ really are intimately related and the following in-depth box explains this relation. We encourage the reader not familiar with this fact to go through this explanation to better understand the profound relation between the T- and Q-operators.

Quantum affine $sl(2)$ and universal operators In many cases, the right way to delve deeper in the core of a theory is to generalise the mathematical setting; this not only opens the way for further achievements but almost always cleans up the table and bring about a great simplification of the structures: complicating to clarify. It turns out that the most natural starting point for the construction of the L- and T-operators is a slight generalisation of the algebra $\mathcal{U}_q\big(su(2)\big)$: the quantum Kač-Moody affine algebra $\mathcal{U}_q\left(\widehat{sl(2)}\right)$. Using this as a starting point we will obtain in one fell swoop a natural description of both T- and Q-operators, displaying explicitly their deep connection, as well as a setting in which the algebraic relations introduced in the previous section can be easily demonstrated. Let us thus introduce the algebra $\mathcal{U}_q\left(\widehat{sl(2)}\right)$: it generated by the six elements $\{x_i, y_i, h_i\}_{i=0}^1$ which satisfy the commutation relations

$$\begin{aligned} [h_i, x_j] &= -a_{i\,j}x_j \ , \quad [h_i, h_j] = 0 \qquad\qquad\ , \ i,j = 0,1 \ , \\ [h_i, y_j] &= a_{i\,j}y_j \quad , \quad [y_i, x_j] = \delta_{i\,j}\tfrac{q^{h_i}-q^{h_j}}{q-q^{-1}} \ , \ i,j = 0,1 \ , \end{aligned}$$

and the quantum Serre relations

$$\begin{aligned} x_i^3x_j - [3]_q x_i^2 x_j x_i + [3]_q x_i x_j x_i^2 - x_j x_i^3 = 0 \ , \\ y_i^3y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0 \quad , \end{aligned}$$

where we define the q-numbers as

$$[n]_q \doteq \frac{q^n - q^{-n}}{q-q^{-1}} \ , \qquad [n]_q \underset{q\to 1}{\longrightarrow} n \ .$$

The matrix $a_{i\,j}$ is the Cartan matrix of the affine algebra $\widehat{su(2)}$:

$$a_{i\,j} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}_{i\,j} \ .$$

In order to be consistently defined this algebra necessitate the further introduction of the grade operator d

$$\begin{aligned} [d, h_0] = [d, h_1] = 0 \ , \quad [d, x_1] = x_1 \ , \\ [d, x_0] = [d, y_0] = 0 \ , \quad [d, y_1] = -y_1 \ , \end{aligned}$$

and the central charge (obviously not the same thing as the central charge c of the CFT!) $k = h_0 + h_1$. The algebra thus defined is a *quasitriangular Hopf algebra* [20, 34, 35] whose co-multiplication

$$\Delta \ : \quad \mathcal{U}_q\left(\widehat{sl(2)}\right) \longrightarrow \mathcal{U}_q\left(\widehat{sl(2)}\right) \otimes \mathcal{U}_q\left(\widehat{sl(2)}\right) \ ,$$

[15] More information on the -matrix approach to integrable models can be found in [33].

is defined by its action on the generators:

$$\Delta(x_i) = x_i \otimes 1 + q^{-h_i} \otimes x_i \ , \quad \Delta(y_i) = y_i \otimes q^{h_i} + 1 \otimes y_i \ ,$$
$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \ , \qquad \Delta(d) = d \otimes 1 + 1 \otimes d \ .$$

There exists a second possible choice for comultiplication:

$$\Delta' \doteq \sigma \circ \Delta \ , \qquad \sigma\left(A \otimes B\right) = B \otimes A \ , \ \forall A, B \in \mathcal{U}_q\left(\widehat{sl(2)}\right) \ ,$$

and the property of quasitriangularity makes sure [36] there exists an object called universal R-matrix

$$\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \ ,$$

intertwining between these two co-multiplications

$$\Delta'(A)\mathcal{R} = \mathcal{R}\Delta(A) \ , \qquad \forall A \in \mathcal{U}_q\left(\widehat{sl(2)}\right) \ . \tag{2.31}$$

Here $\mathcal{B}_+$ and $\mathcal{B}_-$ are the Borel subalgebras of $\mathcal{U}_q\left(\widehat{sl(2)}\right)$ generated by the elements $\{h_0, h_1, y_0, y_1\}$ and $\{h_0, h_1, x_0, x_1\}$, respectively. This universal R-matrix satisfies the Yang-Baxter equation

$$\mathcal{R}^{1\,2}\mathcal{R}^{1\,3}\mathcal{R}^{2\,3} = \mathcal{R}^{2\,3}\mathcal{R}^{1\,3}\mathcal{R}^{1\,2} \ , \qquad \begin{cases} \mathcal{R}^{1\,2} = \mathcal{R} \otimes 1 \\ \mathcal{R}^{2\,3} = 1 \otimes \mathcal{R} \\ \mathcal{R}^{1\,3} = (\sigma \otimes 1)\,\mathcal{R}^{2\,3} \end{cases} ,$$

as is obtained straightforwardly from the action of the comultiplication

$$(\Delta \otimes 1)\,\mathcal{R} = \mathcal{R}^{1\,3}\mathcal{R}^{2\,3} \ , \qquad (1 \otimes \Delta)\,\mathcal{R} = \mathcal{R}^{1\,3}\mathcal{R}^{1\,2} \ ,$$

and the relation (2.31). An explicit form of the universal R-matrix for $\mathcal{U}_q\left(\widehat{sl(2)}\right)$ can be found in [37].

Using the generators introduced above, we can build the following abstract operator

$$\mathcal{L} \doteq e^{i\pi P h}\mathscr{P}\exp\left[\int_0^{2\pi} \mathcal{K}(w)dw\right] \ , \qquad \mathcal{K}(w) \doteq V_-(w)y_0 + V_+(w)y_1 \ ,$$

where the operators P and $V_\pm$ are the same ones we defined in Sect. 2.3 and we set $h_0 = -h_1 = h$, which corresponds to choosing the central charge k to be zero. It is evident that $\mathcal{L} \in \mathcal{B}_+$ and, with some simple computation, one can show that

$$\Delta\left(\mathcal{L}\right) = \left(\mathcal{L} \otimes 1\right)\left(1 \otimes \mathcal{L}\right) \ , \qquad \Delta'\left(\mathcal{L}\right) = \left(1 \otimes \mathcal{L}\right)\left(\mathcal{L} \otimes 1\right) \ .$$

For this reason, the RLL relation follows automatically from the definition of $\mathcal{L}$ and $\mathcal{R}$

$$\mathcal{R}\left(\mathcal{L} \otimes 1\right)\left(1 \otimes \mathcal{L}\right) = \left(1 \otimes \mathcal{L}\right)\left(\mathcal{L} \otimes 1\right)\mathcal{R} \ . \tag{2.32}$$

This is an extremely important relation; being completely abstract, contains in itself the RLL relation for any possible representation of $\mathcal{U}_q\left(\widehat{sl(2)}\right)$. In particular if we are able to map $\mathcal{L}$ into $\mathbf{L}_j$, then the (2.17) will be automatically demonstrated: we need to find an homomorphism between $\mathcal{U}_q\left(\widehat{sl(2)}\right)$ and $\mathcal{U}_q\left(sl(2)\right)$. More precisely we want a whole family of homomorphisms, since we need to introduce the spectral parameter λ which, in this abstract setting, is absent. These homomorphisms actually exist and are named *evaluation representations*:

$$\mathrm{ev}_\lambda \ : \ \mathcal{U}_q\left(\widehat{sl(2)}\right) \to \mathcal{U}_q\left(sl(2)\right) \ , \quad \begin{cases} x_0 \mapsto \lambda^{-1} F\, q^{-\frac{H}{2}} \\ y_0 \mapsto \lambda q^{\frac{H}{2}} E \\ h_0 \mapsto H \end{cases} , \begin{cases} x_1 \mapsto \lambda^{-1} E\, q^{\frac{H}{2}} \\ y_1 \mapsto \lambda q^{-\frac{H}{2}} F \\ h_1 \mapsto -H \end{cases} ,$$

where $\{E, F, H\}$ are the usual generators of $\mathcal{U}_q(sl(2))$. It is immediate to verify that

$$\pi_j(\lambda)[\mathcal{L}] = \mathbf{L}_j(\lambda) \ , \qquad \pi_j(\lambda) \doteq \pi_j \circ \mathrm{ev}_\lambda \ ,$$

while it is less obvious but still verifiable that, starting from the general definition of [37], we obtain

$$(\pi_j(\lambda) \otimes \pi_{j'}(\lambda'))[\mathcal{R}] = \rho_{j\,j'}(\lambda/\lambda')\,\mathbf{R}_{j\,j'}(\lambda/\lambda') \ ,$$

with $\rho_{j\,j'}(\lambda)$ being an uninteresting scalar factor. In this light the operators $\mathbf{T}_j(\lambda)$ are nothing but a specific representation of the following more general "universal T-operator"

$$\mathcal{T} \doteq tr_{\mathcal{U}_q\left(\widehat{sl(2)}\right)}\left[e^{\pi \mathrm{i} P h}\mathcal{L}\right] \ , \tag{2.33}$$

and one can think of defining new operators from $\mathcal{T}$, by choosing different representations of $\mathcal{U}_q\left(\widehat{sl(2)}\right)$. Any two of these operators will commute amongst themselves, as a direct consequence of the universal RLL relation (2.32), and this is precisely a property that we want for our Q-operators. However with the family $\pi_j(\lambda)$, we have exhausted all the finite dimensional evaluation representations: we need to look elsewhere. As mentioned above, the correct choice of algebra for the construction of Q-operators happens to be osc_q. Even though it might look rather different from $\mathcal{U}_q\left(\widehat{sl(2)}\right)$, it is an easy exercise to show that the two following maps

$$\omega_\lambda^\pm \ : \ \mathcal{U}_q\left(\widehat{sl(2)}\right) \to \mathrm{osc}_q \ , \qquad \begin{cases} h \mapsto \pm\mathcal{H} \\ y_0 \mapsto \lambda\mathcal{E}_\pm \\ y_1 \mapsto \lambda\mathcal{E}_\mp \end{cases} \ ,$$

are homomorphisms of $\mathcal{B}_+$ into the quantum oscillator algebra osc_q.

Consider now any representation ρ of osc_q, such that the following object

$$Z(p) \doteq tr_\rho\left[e^{2\pi \mathrm{i} p \mathcal{H}}\right] \ ,$$

exists and do not vanish for $\Im(p) < 0$. Then we can construct the following operators

$$\mathbf{L}_\pm(\lambda) \doteq \rho_\pm(\lambda)[\mathcal{L}] \equiv \rho\left\{e^{\pm\pi\mathrm{i}P\mathcal{H}}\mathscr{P}\exp\left[\lambda\int_0^{2\pi} dw\left(V_-(w)q^{\pm\frac{\mathcal{H}}{2}}\mathcal{E}_\pm + V_+(w)q^{\mp\frac{\mathcal{H}}{2}}\mathcal{E}_\mp\right)\right]\right\} \ ,$$

where

$$\rho_\pm(\lambda) \doteq \rho \circ \omega_\lambda^\pm \ ,$$

and the corresponding realisation of $\mathcal{T}$[16]

$$\mathbf{A}_\pm(\lambda) \doteq Z^{-1}(\pm P) tr_\rho\left[e^{\pm\pi\mathrm{i}P\mathcal{H}}\mathbf{L}_\pm(\lambda)\right] .$$

Notice how, allowing analytic continuation from the $\Im(p) < 0$ half-plane, these operators enjoy the symmetry

$$\mathbf{A}_\pm(\lambda) \underset{(p,\varphi(w))\to(-p,-\varphi(w))}{\longrightarrow} \mathbf{A}_\mp(\lambda) \ .$$

These operators, as much as the other we introduced, have to be understood as power series in λ^2 (since, here too, the odd terms vanish under the trace)

$$\mathbf{A}_\pm(\lambda) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{\{\sigma_i=\pm 1\}_{i=1}^{2n} \\ \sum_i \sigma_i = 0}} \lambda^{2n} a_{2n}(\sigma_1, \ldots, \sigma_{2n}| \pm P)\, J_{2n}(\mp\sigma_1, \mp\sigma_2, \ldots, \mp\sigma_{2n}) \ , \tag{2.34}$$

[16] Note that in this case we need to add a regularising factor to the trace. This is due to infinite dimensionality of the algebra osc_q.

where we introduced the two functions

$$J_{2n}(\sigma_1,\dots,\sigma_{2n}) \doteq q^n \int_{\substack{0\\ w_1\geq\cdots\geq w_{2n}}}^{2\pi} dw_1\cdots dw_{2n} V_{\sigma_1}(w_1)\cdots V_{\sigma_{2n}}(w_{2n}) \ ,$$

$$a_{2n}(\sigma_1,\dots,\sigma_{2n}|P) \doteq Z^{-1}(P) tr_\rho \left[e^{2\pi \mathrm{i} P\mathcal{H}}\mathcal{E}_{\sigma_1}\cdots\mathcal{E}_{\sigma_{2n}}\right] \ .$$

The really interesting fact about this decomposition is that the dependance on $\mathcal{U}_q\left(\widehat{sl(2)}\right)$ is contained entirely in the coefficients a_{2n} which turn out to be uniquely determined by the commutation relations of the generators and the cyclic properties of the trace; as such these do not depend on the chosen representation ρ of osc_q! Clearly the coefficients J_{2n} are closely related to the non-local integrals of motion $\mathbf{G}_{2n}$, however a more neat expansion of the operators $\mathbf{A}_\pm$ is the following

$$\log\left(\mathbf{A}_\pm(\lambda)\right) = -\sum_{n=1}^{\infty} y^{2n}\mathbf{H}_{2n} \ , \qquad y \doteq \frac{\Gamma\left(1-\beta^2\right)}{\beta^2}\lambda \ , \tag{2.35}$$

where the $\mathbf{H}_{2n}$ are a set of non-local integrals of motion alternative to the $\mathbf{G}_{2n}$ ones and, obviously, algebraically related to these, e.g.

$$\mathbf{H}_1 = \frac{\beta^4\Gamma\left(\beta^2\right)}{4\pi\Gamma\left(1-\beta^2\right)\sin\left(2\pi P+\pi\beta^2\right)}\mathbf{G}_1 \ . \tag{2.36}$$

Finally we can introduce the Baxter Q operators as

$$\mathbf{Q}_\pm(\lambda) \doteq \lambda^{\pm 2\frac{P}{\beta^2}}\mathbf{A}_\pm(\lambda) \ .$$

Just as the operators $\mathbf{T}_j(\lambda)$, they act invariantly on $\mathscr{F}_p$, which is an immediate consequence of the representation (2.34) of $\mathbf{A}_\pm$. Below is a list of the properties of Q operators, which descend from from their definition as representations of $\mathcal{T}$ (2.33) and from the structure of the representations involved:

1. they commute amongst themselves and with all the T-operators

$$[\mathbf{Q}_\pm(\lambda),\mathbf{Q}_\pm(\lambda')] = [\mathbf{Q}_\pm(\lambda),\mathbf{Q}_\mp(\lambda')] = [\mathbf{Q}_\pm(\lambda),\mathbf{T}_j(\lambda')] = 0 \ .$$

   Consequently they commute with all the IMs, local and nonlocal

$$[\mathbf{Q}_\pm(\lambda),\mathbf{I}_{2k-1}] = [\mathbf{Q}_\pm(\lambda),\mathbf{G}_{2n}] = [\mathbf{Q}_\pm(\lambda),\mathbf{H}_{2n}] = 0 \ ;$$

2. they satisfy the Baxter $T-Q$ relation

$$\mathbf{T}(\lambda)\mathbf{Q}_\pm(\lambda) = \mathbf{Q}_\pm(q\lambda) + \mathbf{Q}_\pm(q^{-1}\lambda) \ .$$

   This relation is a second order finite-difference equation whose "potential" $\mathbf{T}(\lambda)$ is a periodic function of $\log\left(\lambda^2\right)$; for this reason the two solutions $\mathbf{Q}_+$ and $\mathbf{Q}_-$ can be interpreted as Bloch wave solutions to the $T-Q$ relation.

3. they satisfy the quantum wronskian relation

$$\mathbf{Q}_+(q^{\frac{1}{2}}\lambda)\mathbf{Q}_-(q^{-\frac{1}{2}}\lambda) - \mathbf{Q}_+(q^{-\frac{1}{2}}\lambda)\mathbf{Q}_-(q^{\frac{1}{2}}\lambda) = 2\mathrm{i}\sin\left(2\pi P\right) \ . \tag{2.37}$$

   This relation guarantees the independence of the functions $\mathbf{Q}_+$ and $\mathbf{Q}_-$, solutions to the $T-Q$ relation

4. Wronskian expression of $\mathbf{T}_j$

$$2\mathrm{i}\sin\left(2\pi P\right)\mathbf{T}_j(\lambda) = \mathbf{Q}_+(q^{j+\frac{1}{2}}\lambda)\mathbf{Q}_-(q^{-j-\frac{1}{2}}\lambda) - \mathbf{Q}_+(q^{-j-\frac{1}{2}}\lambda)\mathbf{Q}_-(q^{j+\frac{1}{2}}\lambda) \ . \tag{2.38}$$

The asymptotic of $\mathbf{A}_\pm$: a simple case A particularly simple situation is when $2p = N$, for some integer N; then the quantum wronskian (2.37) vanishes, meaning that the Q-functions $\mathbf{Q}_+$ and $\mathbf{Q}_-$ are linearly dependent. In particular rather simple direct computation shows that $\mathbf{A}_+(\lambda)\Big|_{2p=N} = \mathbf{A}_-(\lambda)\Big|_{2p=N} = \mathbf{A}^{(N)}(\lambda)$, where this last operator can be written as

$$\mathbf{A}^{(N)}(\lambda) \doteq \sum_{n=0}^{\infty} \frac{\varkappa^{2n}}{(n!)^2} q^n \mathscr{P}_w \left\{ \left[\int_0^{2\pi} \frac{dw}{2\pi} V_+(w) \right]^n \left[\int_0^{2\pi} \frac{dw}{2\pi} V_-(w) \right]^n \right\} \Bigg|_{2p=N} ,$$

with

$$\varkappa \doteq -\mathrm{i} \frac{\pi\lambda}{\sin(\pi\beta^2)} ,$$

and the symbol $\mathscr{P}_w$ orders the factors from left to right in decreasing w order. The vacuum eigenvalue of this operator $\mathbf{A}^{(N)}(\lambda) \left| p = \frac{N}{2} \right\rangle = A_N^{(\mathrm{vac})}(\lambda) \left| p = \frac{N}{2} \right\rangle$ coincides with the one-dimensional Coulomb gas partition function

$$A_N^{(\mathrm{vac})}(\lambda) \equiv \mathscr{Z}_N(\varkappa) \doteq \sum_{n=0}^{\infty} \frac{\varkappa^{2n}}{(n!)^2} \int_0^{2\pi} \left(\prod_{\ell=1}^{n} \frac{dw_\ell d\tilde{w}_\ell}{4\pi^2} \right) e^{\mathrm{i}N \sum_{\ell=1}^{n} (\tilde{w}_\ell - w_\ell)} \times$$
$$\times \frac{\prod_{i \neq j} \left| 4 \sin\left(\frac{w_i - w_j}{2}\right) \sin\left(\frac{\tilde{w}_i - \tilde{w}_j}{2}\right) \right|^{2\beta^2}}{\prod_{i,j} \left| 2 \sin\left(\frac{w_i - \tilde{w}_j}{2}\right) \right|^{2\beta^2}} ,$$

which can be shown [22] to define an entire function of $\varkappa^2$, with asymptotic behaviour

$$\log\left(\mathscr{Z}_N(\varkappa)\right) \underset{\varkappa^2 \to \infty}{\sim} \varkappa^{\frac{1}{1-\beta^2}} .$$

In fact with some effort one can show that the same beaviour is valid for all the eigenvalues of $\mathbf{A}^{(N)}(\lambda)$, meaning that

$$\log\left(\mathbf{A}_\pm(\lambda)\right)\Big|_{2p=N} \underset{\lambda^2 \to -\infty}{\sim} M \left(-\lambda^2\right)^{\frac{1}{2-2\beta^2}} . \tag{2.39}$$

Although the entirety in λ^2 of operators $\mathbf{A}_\pm(\lambda)$ can be demonstrated in general, the above asymptotic behavior is explicitly demonstrated only for $2p = N \in \mathbb{Z}$. Nonetheless it is reasonable to assume that this behaviour is valid for any value of p as we shall do.

2.6 Bethe ansatz and non-linear integral equation

In this section we will concentrate on the eigenvalue $Q(\lambda) \equiv Q_+^\alpha(\lambda)$ of $\mathbf{Q}_+(\lambda)$ on the state $|\alpha\rangle \in \mathscr{F}_p$; similar considerations can be obtained for the eigenvalues of $\mathbf{Q}_-(\lambda)$.

Let us denote $T(\lambda)$ and $A(\lambda)$ the eigenvalues of $\mathbf{T}(\lambda)$ and $\mathbf{A}_+(\lambda)$, respectively, on the state $|\alpha\rangle$. Then the following two equations descend directly from Baxter $T - Q$ relation

$$T(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda) ,$$
$$T(\lambda)A(\lambda) = e^{2\pi i p} A(q\lambda) + e^{-2\pi i p} A(q^{-1}\lambda) .$$

As we will shortly see, provided the analytic properties of the functions A and T, these equations impose severe restrictions on the allowed solutions. Let us recall the properties of A and T we agree on

- Analyticity: both functions $A(\lambda)$ and $T(\lambda)$ are entire in $\lambda^2 \in \mathbb{C}$;

- Asymptotic behaviour:

$$A(\lambda) \underset{\lambda^2\to-\infty}{\sim} \exp\left[M\left(-\lambda^2\right)^{\frac{1}{2-2\beta^2}}\right] ,$$
$$T(\lambda) \underset{\substack{|\lambda^2|\to\infty \\ |\arg(\lambda^2)|<\pi}}{\sim} \exp\left[m\left(\lambda^2\right)^{\frac{1}{2-2\beta^2}}\right] ,$$

  where m is given by the formula (2.22) and M is presented below in (2.42).

- Location of zeroes[17]: the zeroes $\left\{\lambda_k^2\right\}_{k=0}^{\infty}$ of $A(\lambda)$ are either real or pairs of complex conjugates. For any eigenvalue $A(\lambda)$, the number of zeroes on the positive real λ^2-axis accumulate towards $+\infty$, while the number of other zeroes remains finite. For the vacuum, if $2p > -\beta^2$, the only zeroes are those on the positive real λ^2-axis. We avoid those values of the highest weight (e.g. $2p = -\beta^2 - n$) for which $\lambda_0^2 = 0$.

These properties allow us to use Hadamard factorisation theorem: if $0 < \beta^2 < \frac{1}{2}$ then the asymptotic behaviour of $A(\lambda)$ tells us that its order ρ_A, as a function λ^2, is $\frac{1}{2} < \rho_A < 1$, meaning that we can write the very simple product

$$A(\lambda) = \prod_{k=0}^{\infty}\left(1 - \frac{\lambda^2}{\lambda_k^2}\right) , \qquad A(0) = 1 .$$

Now, let us take the "$T - A$ relation", which we can write as

$$e^{2\pi \mathrm{i} p} T(\lambda)\frac{A(\lambda)}{A(q^{-1}\lambda)} = \mathfrak{a}(\lambda) + 1 , \qquad \mathfrak{a}(\lambda) \doteq e^{4\pi \mathrm{i} p}\frac{A(q\lambda)}{A(q^{-1}\lambda)} ,$$

and evauate it at $\lambda^2 = \lambda_k^2$, recalling that $T(\lambda)$ is devoid of singularities at finite λ^2; what we obtain is an infinite set of coupled algebraic equations of Bethe ansatz-type[18]

$$\mathfrak{a}(\lambda_k) = -1 \quad \Longrightarrow \quad \prod_{\ell=0}^{\infty}\frac{\lambda_\ell^2 - q^2\lambda_k^2}{\lambda_\ell^2 - q^{-2}\lambda_k^2} = -e^{-4\pi \mathrm{i} p} , \qquad \forall k \in \mathbb{N}^0 .$$

As always the infinity of equations and variables at hand might sound slightly scary, however there exists a beautiful procedure which allow us to "resum" these equations turning them into a single non-linear integral equation (NLIE) paired with a finite set of Bethe ansatz equations for a finite set of variables. The introduction of this method in the context of QFT is due to C. Destri and H.J. de Vega [39], although the non-linear integral equation first appeared in a work of A. Klümper, M.T. Batchelor and P.A. Pearce [40], where it was used to compute the central charge of 6- and 19-vertex models. We will not present the derivation of the equations, as it follows the same exact lines of the original article; we limit ourselves to displaying the result:

$$\begin{cases} \mathrm{i}\log\left(\mathfrak{a}(\theta)\right) = -2\pi\frac{p}{\beta^2} + 2M\cos\left(\pi\frac{\beta^2}{2-2\beta^2}\right)e^{\theta} + \\ \qquad\qquad +\mathrm{i}\sum_a \log\left(S(\theta-\theta_a)\right) - 2\mathcal{G}\star\Im\left[\log\left(1+\mathfrak{a}(\theta-\mathrm{i}0)\right)\right] \\ \mathfrak{a}(\theta_a) = -1 \end{cases} , \tag{2.40}$$

where, as before, $\lambda^{1+\xi} \equiv \lambda^{\frac{1}{1-\beta^2}} = e^{\theta}$ and $\{\theta_a\}_a$ corresponds to the set of those zeroes $\left\{\lambda_a^2 = e^{2\theta_a(1-\beta^2)}\right\}_a$ which lie outside the real positive λ^2-axis. We used the sign $\star$ to denote the convolution of two functions:

$$f\star g(\theta) \doteq \int_{-\infty}^{\infty} d\theta' f(\theta-\theta')g(\theta') \equiv \int_{-\infty}^{\infty} d\theta' f(\theta')g(\theta-\theta') ,$$

and we introduced the kernel

$$\mathcal{G}(\theta) \doteq \delta(\theta) + \frac{1}{2\pi\mathrm{i}}\partial_\theta \log\left(S(\theta)\right) ,$$

[17] We will not make use of the knowledge about the zeroes of $T(\lambda)$.
[18] For more information on Bethe Ansatz method of solution for integrable models see [38]

and the function

$$S(\theta) \doteq \exp\left[-\mathsf{i}\int_{-\infty}^{\infty}\frac{d\nu}{\nu}\sin(\nu\theta)\,\frac{\sinh\left(\pi\nu\frac{1+\xi}{2}\right)}{\cosh\left(\pi\frac{\nu}{2}\right)\sinh\left(\pi\nu\frac{\xi}{2}\right)}\right] ,$$

which coincides precisely with the soliton-soliton scattering amplitude for the sine-Gordon model [41]. Given a solution $\mathfrak{a}(\lambda)$ of the NLIE (2.40) above, one can recover the function $A(\lambda)$ with the following formula

$$\log(A(\lambda)) = -\mathsf{i}\int_{-\infty-\mathsf{i}}^{\infty-\mathsf{i}} d\nu \frac{g(\nu+\mathsf{i}0)}{\cosh\left(\pi\frac{\nu+\mathsf{i}0}{2}\right)\sinh\left(\pi\xi\frac{\nu+\mathsf{i}0}{2}\right)}\left(-\lambda^2\right)^{\mathsf{i}\nu\frac{1+\xi}{2}} ,$$

$$g(\nu) = \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\Im\left[\log\left(1+\mathfrak{a}(\theta-\mathsf{i}0)\right)\right]e^{-\mathsf{i}\nu\theta} . \tag{2.41}$$

It is possible to use this formulae in the case of the vacuum eigenvalue $A^{(\mathrm{vac})}(\lambda)$ in the limit $p\to\infty$ to compute the exact form of the coefficient M in (2.39). This turns out to be

$$M = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\xi}{2}\right)\Gamma\left(\frac{1-\xi}{2}\right)\left(\Gamma\left(\frac{1}{1+\xi}\right)\right)^{1+\xi} . \tag{2.42}$$

We will not present the computations here and refer the interested reader to the original article [2].

3 Integrable structures of massive integrable field theories

Having extracted and analysed the integrable structures of conformal field theories, a natural question arises: are these results "exportable" in massive field theories? The answer, at least for what concerns theories obtained as integrable deformations of CFTs, is positive. Actually, as it turns out, this extension is rather straightforward: the equations keep the same exact form they have in the massless case. As we already noticed above, this is expected since the algebraic structure governing massless theories survives unscathed to the integrable deformation. On the other hand, the analytic properties of the various objects we introduced undergo a radical change as a consequence of the interplay between the two chiralities which, in presence of a mass scale, is no more trivial.

In the following we will first briefly review A.B. Zamolodchikov results concerning integrable deformations of CFTs [42] and then construct the T-operators for a particular class of these.

3.1 Brief overview of CFT integrable deformations

Remember how in a CFT there exists an infinite set of integrals of motion, which can be constructed from normal ordered products of the energy momentum tensor $T(u)$ and its derivatives:

$$\mathbf{I}_{2k-1} = \int_0^{2\pi}\frac{dw}{2\pi}T_{2k}(w) , \qquad \overline{\mathbf{I}}_{2k-1} = \int_0^{2\pi}\frac{d\overline{w}}{2\pi}\overline{T}(\overline{w}) .$$

All these IMs are in involution: they form an abelian subalgebra $\mathcal{I}$ of $\mathcal{U}(\mathrm{Vir})$ (same goes for the left chirality, obviously).

Given a CFT with Hamiltonian H_{CFT}, we can think of deforming it by a relevant field Φ (clearly belonging to that same CFT), obtaining thus a massive field theory

$$H_\Phi \doteq H_{\mathrm{CFT}} + \hat{\mu}^2\int\Phi(x)dx^2 .$$

In general these theories are not integrable: the appearance of a mass scale inevitably destroys the conformal symmetry and, along with it, the abelian subalgebra containing the IMs. Only some of these survive and can be written as

$$\mathbb{I}_s = \int_0^{2\pi}\left[T_{s+1}dw + \Theta_{s-1}d\overline{w}\right] , \qquad \overline{\mathbb{I}}_s = \int_0^{2\pi}\left[\overline{\Theta}_s dw + \overline{T}_{s+1}d\overline{w}\right] ,$$

where T_s, $\overline{T}_s$, Θ_s and $\overline{\Theta}_s$ are some local fields satisfying the current conservation law

$$\overline{\partial} T_{s+1} = \partial \Theta_{s-1} \ , \qquad \partial \overline{T}_{s+1} = \overline{\partial} \overline{\Theta}_{s-1} \ ,$$

and the index s takes values in a finite set: $s \in \mathcal{X}$, $\mathfrak{C}(\mathcal{X}) < \infty$, where $\mathfrak{C}$ denotes the cardinality of a set. It turns out, however, that with the right choice of perturbing field this set $\mathcal{X}$ becomes infinite; in other words there exist particular perturbations for which the abelian subalgebra $\mathcal{I}$ survives in its entirety[19]. As a consequence the massive theory obtained through these deformations is integrable inheriting *in toto*, with suitable modifications, the integrable structure of the corresponding CFT. In [42] Al.B. Zamolodchikov showed[20] that this phenomenon happens in CFTs with $c < 1$ if one chooses $\Phi_{(1,3)}$, $\Phi_{(1,2)}$ or $\Phi_{(2,1)}$ as perturbations, where the bracketed indices stands for (r, s), identifying the fields on the Kač Table [8]. In the following we will concentrate on $\Phi_{(1,3)}$ perturbations only.

The $\Phi_{(1,3)}$ perturbations of CFT Consider the massive field theory defined by the Hamiltonian

$$H_{(1,3)} \doteq H_{\text{CFT}} + \hat{\mu}^2 \int \Phi_{(1,3)}(x) dx^2 \ ,$$

where $\Phi_{(1,3)}$ is a primary field of H_{CFT} with conformal dimensions

$$h_{1,3} = \overline{h}_{1,3} = 2\beta^2 - 1 \equiv \frac{\xi - 1}{\xi + 1} \ ,$$

which satisfies the null-vector equations

$$\left[2(1 + 2\beta^2) L_{-3} - 4 L_{-1} L_{-2} + \frac{1}{\beta^2} L_{-1}^3 \right] \Phi_{(1,3)}(w, \overline{w}) = 0 \ , \tag{3.1}$$

$$\left[2(1 + 2\beta^2) \overline{L}_{-3} - 4 \overline{L}_{-1} \overline{L}_{-2} + \frac{1}{\beta^2} \overline{L}_{-1}^3 \right] \Phi_{(1,3)}(w, \overline{w}) = 0 \ , \tag{3.2}$$

and is assumed to have the following canonical normalisation

$$\left\langle \Phi_{(1,3)}(w, \overline{w}) \, \Phi_{(1,3)}(w', \overline{w}') \right\rangle_{\text{CFT}} \underset{(w,\overline{w}) \to (w', \overline{w}')}{\sim} \left| w - w' \right|^{-4 h_{1,3}} .$$

From dimensional analysis one immediately sees that $\left[\hat{\mu}^2\right] = [\text{length}]^{h_{1,3}-1}$ and $\hat{\mu}$ can be thought of carrying an anomalous dimension of $h_{\hat{\mu}} = \overline{h}_{\hat{\mu}} = \frac{1 - h_{1,3}}{2} = \frac{1}{1+\xi}$. Notice how this is exactly the opposite of the anomalous dimension carried by the spectral parameter λ in the analysis of the above section.

Any field theory is completely characterised by the infinite-dimensional vector space of local fields $\Omega = \{\mathcal{O}_j\}_{j \in \mathbb{N}}$, along with the totality of their correlation functions. Generically, when taking a perturbation of a CFT, the fields of this last require an infinite number of renormalisation parameters in order to cancel the ultraviolet divergencies which arise in correlation functions. In the case under consideration, however, the massive theory turns out to be "super-renormalisable", meaning only a finite number of counter-terms is needed (for $\beta^2 < \frac{1}{2}$, that is $c < -2$, there are actually no UV divergencies at all). If, furthermore, we are in a finite size geometry, as we are, the infrared divergencies are completely under control, thanks to the natural cutoff $R < \infty$. For these reasons we can safely assume that the Hilbert space of the perturbed theory has the same structure of that of the original CFT

$$\mathscr{H}_{(1,3)} \simeq \mathscr{H}_{\text{CFT}} \ , \tag{3.3}$$

and there exists a one-to-one correspondence between fields of the two theories. In other words one can assign to fields in the massive theory the roles they played in the CFT; in this sense there exists in the massive theory a concept of primary fields, of descendants and also of Virasoro operators.

Consider the subspace $\Lambda_{\text{CFT}} \subset \Omega_{\text{CFT}}$, consisting of all the composite fields built out of $T(w)$ and its derivatives. Now take the quotient of this subspace by the action of L_{-1}:

$$\hat{\Lambda}_{\text{CFT}} \doteq \Lambda_{\text{CFT}} \Big/ \left(L_{-1} \Lambda_{\text{CFT}} \right) \subset \Lambda_{\text{CFT}} \ .$$

[19] Actually this is not strictly a consequence of $\|\mathcal{X}\| = \infty$, but for the models we are going to analyse it is conjectured to be so.

[20] Actually he made conjectures based on strong physical assumptions. His conjectures were never disproved since then and are assumed to be true.

This further subset $\hat{\Lambda}_{\text{CFT}}$ consists of all those composite fields which are not total derivatives (remember that L_{-1} acts on local fields as a derivative). Then it is immediate to notice that all the integrals of motion are generated as integrals of some element of $\hat{\Lambda}_{\text{CFT}}$; informally

$$\mathcal{I} = \int \hat{\Lambda}_{\text{CFT}} \ .$$

Now, while clearly $\overline{\partial}\hat{\Lambda}_{\text{CFT}} = 0$, this is no more true when considering the corresponding subspace in the deformed theory; in general one will have

$$\overline{\partial} T_s = \sum_{n=1}^{\infty} \mu^{2n} R_{s-1}^{(n)} \ , \qquad T_s \in \Lambda \ , \qquad R_{s-1}^{(n)} \in \Omega \ .$$

By simple dimensional analysis, we see that the fields $R_{s-1}^{(n)}$ must have dimensions $(h, \overline{h}) = (s - n + nh_{1,3}, 1 - n + nh_{1,3})$; however, no field in Ω is allowed to have negative left conformal dimension, meaning that the series above must truncate[21] for n bigger than some integer $N \geq 1$. Moreover, since $\Phi_{(1,3)}$ is the most relevant field in its OPE subalgebra, we conclude that $N = 1$

$$\overline{\partial} T_s = \hat{\mu}^2 R_{s-1} \ .$$

Denoting as usual with $\mathcal{V}_{1,3}$ the Verma module with highest weight $h_{1,3}$, we have

$$L_0 \mathcal{V}_{1,3}^{(s)} = (h_{1,3} + s)\, \mathcal{V}_{1,3}^{(s)} \ , \quad \overline{L}_0 \mathcal{V}_{1,3}^{(s)} = h_{1,3} \mathcal{V}_{1,3}^{(s)} \ , \qquad \mathcal{V}_{1,3} = \bigoplus_{s=0}^{\infty} \mathcal{V}_{1,3}^{(s)} \ ,$$

and, clearly[22]

$$R_{s-1} \in \mathcal{V}_{1,3}^{(s-1)} \ .$$

We can thus interpret the anti-holomorphic partial derivative $\overline{\partial}$ as a map between subspaces of Ω:

$$\overline{\partial} \ : \quad \hat{\Lambda}_s \longrightarrow \mathcal{V}_{1,3}^{(s-1)} \ .$$

By taking first-order corrections to correlation functions involving he field T_s, one can show that

$$\overline{\partial} T_s(w, \overline{w}) = \left[T_s(w, \overline{w}) \, , \, \hat{\mu}^2 \int_0^{2\pi} dw' \Phi_{(1,3)}(w', \overline{w}) \right] \ ,$$

which is a usual formula of perturbation theory. As a consequence

$$[\overline{\partial} \, , \, L_{-1}] = 0 \ ,$$

and we can define a set of operators $D_n \ : \ \Lambda \to \mathcal{V}_{1,3}$ from their action on the vectors of Λ, informally:

$$D_n \Lambda \doteq \left[\Lambda \, , \, \hat{\mu}^2 \int_0^{2\pi} dw' e^{in(w'-w)} \Phi_{(1,3)}(w', \overline{w}) \right] \ .$$

Clearly we have $\overline{\partial} \equiv D_0$ and it is fairly easy to prove the following relations

$$\begin{aligned} [L_n \, , \, D_m] &= -\left[(1 - h_{1,3})(n+1) + m \right] D_{n+m} \ , \\ D_{-n-1} \cdot 1 &= \frac{1}{n!} L_{-1}^n \Phi_{(1,3)}(w, \overline{w}) \ . \end{aligned}$$

[21] Remember that $h_{1,3} = 2\beta^2 - 1$ and, if $c < 1$ then $h_{1,3} < 1$.

[22] Note that we care-freely apply the CFT concept of Verma modules to the deformed theory. This can actually be done thanks to the isomorphism (3.3), since we logically expect that the decomposition of the CFT Hilbert space into Verma modules survives to the deformation along with the other structures. Therefore it should exist in the massive theory a decomposition of $\mathscr{H}_{1,3} = \bigoplus_a \left(\mathscr{V}_a \otimes \overline{\mathscr{V}_a} \right)$ into spaces $\mathscr{V}_a \simeq \mathcal{V}_a$. We use the same notation as in the CFT, hoping this note will be sufficient to avoid confusion.

Finally we are at a point where we can explicitly compute the action of $\overline{\partial}$ on the elements of $\hat{\Lambda}$. Let us begin with the energy-momentum tensor $T_2 \equiv T = L_{-2}\cdot 1$ itself

$$\overline{\partial}T = \hat{\mu}^2 D_0 L_{-2}\cdot 1 = \hat{\mu}^2(h_{1,3}-1)D_{-2}\cdot 1 = \hat{\mu}^2(h_{1,3}-1)L_{-1}\Phi_{(1,3)} \ .$$

Introducing the local field $\Theta_0 \equiv \Theta \doteq \hat{\mu}^2(h_{1,3}-1)\Phi_{(1,3)}$, we see that we can write

$$\overline{\partial}T_2 = \partial\Theta_0 \ ,$$

which is a current conservation equation and, as such, define an integral of motion $\mathbb{I}_1$.

Let us now try and see if there's a current conservation law also for the next element of $\hat{\Lambda}$: $T_4 \equiv L_{-2}^2\cdot 1$. In this case we obtain

$$\begin{aligned}\overline{\partial}T_4 &= \hat{\mu}^2 D_0 L_{-2}^2\cdot 1 = \hat{\mu}^2(h_{1,3}-1)\left(D_{-2}L_{-2}+L_{-2}D_{-2}\right)\cdot 1\\ &= \hat{\mu}^2(h_{1,3}-1)\left(2L_{-2}L_{-1}+\frac{h_{1,3}-3}{6}L_{-1}^3\right)\Phi_{(1,3)} \ ,\end{aligned}$$

and we cannot write $\overline{\partial}T_4$ as the holomorphic derivative of a local field! As a consequence there seem to be no conservation law.

Degenerate fields and integrals of motion We have seen that there seems to be no hope of recovering the continuity equations $\overline{\partial}T_{2n} = \partial\Theta_{2n-2}$ for $n>1$ and, as such, we have only two integrals of motion: $\mathbb{I}_1$ and $\overline{\mathbb{I}}_1$, which is to say the energy $\mathbb{I}_1+\overline{\mathbb{I}}_1$ and the momentum $\mathbb{I}_1-\overline{\mathbb{I}}_1$ of our system. However, we forgot that we have an ace in our sleeve: we chose the perturbing field to be $\Phi_{(1,3)}$, a degenerate field which satisfies the level-3 null vector equations (3.1) and (3.2)! Recalling that $L_{-2}L_{-1} = L_{-1}L_{-2}-L_{-3}$, we can rewrite $\overline{\partial}T_4 = \partial\Theta_2$ where

$$\Theta_2 \doteq \hat{\mu}^2\frac{h_{1,3}-1}{h_{1,3}+2}\left(2h_{1,3}L_{-2}+(h_{1,3}-2)\,\frac{(h_{1,3}-1)\,(h_{1,3}+3)}{6\,(h_{1,3}+1)}L_{-1}^2\right)\Phi_{(1,3)} \ ,$$

which is a proper conservation law and give rise to the integral of motion $\mathbb{I}_3$.

The conjecture of Al. Zamolodchikov is that the phenomenon illustrated just above, happens on every level subspace $\mathcal{V}_{(1,3)}^{(s-1)}$ with odd s. A nice way to see this for $s\leq 7$ is the following. Consider the operator B_s defined as

$$B_s \doteq \Pi_{s-1}\overline{\partial} \ : \quad \hat{\Lambda}_s \longrightarrow \hat{\mathcal{V}}_{(1,3)}^{(s-1)} \ , \qquad \hat{\mathcal{V}}_{(1,3)}^{(s-1)} \doteq \mathcal{V}_{(1,3)}^{(s-1)} \Big/ \left(L_{-1}\mathcal{V}_{(1,3)}^{(s-1)}\right) \ ,$$

where Π_s is the projector onto $\hat{\mathcal{V}}_{(1,3)}^{(s-1)}$:

$$\Pi_s \ : \quad \mathcal{V}_{(1,3)}^{(s-1)} \longrightarrow \hat{\mathcal{V}}_{(1,3)}^{(s-1)} \ .$$

By definition, if $B_{s+1}T_{s+1}=0$, then we are assured that there exists a field Θ_{s-1} such that $\overline{\partial}T_{s+1} = \partial\Theta_{s-1}$. This means that a conservation law is present at the level s iff B_s has a non-vanishing kernel. The two conservation laws we found above appear for a very simple reason

$$\dim\left(\hat{\mathcal{V}}_{(1,3)}^{(1)}\right) = \dim\left(\hat{\mathcal{V}}_{(1,3)}^{(3)}\right) = 0 \ .$$

This suggests that we can try to compare the dimensions of the spaces $\hat{\Lambda}_s$ and $\hat{\mathcal{V}}_{(1,3)}^{(s-1)}$; we can do this by using the character formulae [9]:

$$\sum_{s=0}^{\infty} q^s \dim\left(\hat{\mathcal{V}}_{(1,3)}^{(s-1)}\right) = (1-q)\chi_{(1,3)}(q) \ , \qquad \chi_{(1,3)}(q) \doteq \prod_{\substack{n=1\\ n\neq 3}}^{\infty}(1-q^n)^{-1} \ ,$$

$$\sum_{s=0}^{\infty} q^s \dim\left(\hat{\Lambda}_s\right) = (1-q)\chi_0(q)+q \ , \qquad \chi_0(q) \doteq (1-q)\prod_{n=2}^{\infty}(1-q^n)^{-1}+q \ .$$

In the following table we list the first few dimensions

s	1	2	3	4	5	6	7	8	9	10	11
$\dim\left(\hat{\mathcal{V}}^{(s-1)}_{(1,3)}\right)$	1	0	1	0	2	0	3	1	4	2	7
$\dim\left(\hat{\Lambda}_s\right)$	0	1	0	2	1	3	2	5	4	8	7

We see that for s odd and $s \leq 7$, $\dim\left(\hat{\Lambda}_s\right) = \dim\left(\hat{\mathcal{V}}^{(s-1)}_{(1,3)}\right) + 1$, meaning that B_{s+1} has always at least a one-dimensional kernel assuring the conservation laws. As s grows over 7, however, this argument fails and the existence of higher spin conservation laws is a conjecture which finds a partial justification in the classical limit (where their existence is guaranteed to be true).

As a last note, one should pay particular attention when c takes values corresponding to the minimal models $\mathcal{M}_{p,p'}$, as in those cases the structure of the spaces $\hat{\Lambda}_s$ and $\hat{\mathcal{V}}^{(s-1)}_{(1,3)}$ is warped by the infinite ladder of null-vectors. Moreover in a minimal model, it not justified to assume that $\overline{\partial} T_s = \hat{\mu}^2 R_{s-1}$: terms with higher powers of $\hat{\mu}^2$ might appear.

3.2 T-operators in $\Phi_{(1,3)}$ deformed CFTs

Let us consider the theory defined by the following formal action

$$\mathscr{A}_{(1,3)} \doteq \mathscr{A}_{\text{CFT}} + \hat{\mu}^2 \int du dv \Phi_{(1,3)}(u,v) \ , \qquad [\hat{\mu}] = [\text{length}]^{2\beta^2-2} \ , \tag{3.4}$$

defined on a cylinder of radius[23] R: $\left\{(u,v) \;\middle|\; u+R=u\right\}$, with u playing the role of space and v that of time. Here $\mathscr{A}_{\text{CFT}}$ is the formal action of a conformal field theory with $c<1$ and $\Phi_{(1,3)}$ is its primary field of conformal dimension $h_{1,3} = 2\beta^2 - 1$. We recall the relation between β and the central charge:

$$\beta = \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}} \implies c = 13 - 6\left(\beta^2 + \beta^{-2}\right) \ .$$

As we already mentioned above, important facts of deformed CFTs are the presence of a mass scale $\mathfrak{m} \propto \hat{\mu}^{\frac{1}{1-\beta^2}}$ and the breaking of conformal invariance, which translates into the non-holomorphicity of the energy-momentum tensor. As we have seen above, however, in a $\Phi_{(1,3)}$-deformed CFT, the local fields $T_{2k}(w,\overline{w})$ satisfy the continuity equations

$$\overline{\partial} T_{2k}(w,\overline{w}) = \partial \Theta_{2k-2}(w,\overline{w}) \ , \qquad \partial \overline{T}_{2k}(w,\overline{w}) = \overline{\partial \Theta}_{2k-2}(w,\overline{w}) \ ,$$

meaning that the integrals

$$\mathbb{I}_{2k-1} \doteq \int_0^R \frac{du}{2\pi} \left[T_{2k}(w,\overline{w}) + \Theta_{2k-2}(w,\overline{w})\right] \ , \tag{3.5}$$

$$\overline{\mathbb{I}}_{2k-1} \doteq \int_0^R \frac{du}{2\pi} \left[\overline{T}_{2k}(w,\overline{w}) + \overline{\Theta}_{2k-2}(w,\overline{w})\right] \ , \tag{3.6}$$

do not depend on the "time" v and are, as such, integrals of motion. It is not hard to verify, at least for small k, that these integrals are in involution

$$[\mathbb{I}_{2k-1}, \mathbb{I}_{2l-1}] = [\mathbb{I}_{2k-1}, \overline{\mathbb{I}}_{2l-1}] = [\overline{\mathbb{I}}_{2k-1}, \overline{\mathbb{I}}_{2l-1}] = 0 \ .$$

Notice that $\mathbb{H} \doteq \mathbb{I}_1 + \overline{\mathbb{I}}_1$ and $\mathbb{P} = \mathbb{I}_1 - \overline{\mathbb{I}}_1$ are, respectively, the Hamiltonian and the momentum operators of $\mathscr{A}_{(1,3)}$.

[23] We slightly change the geometry here by introducing the radius as a new parameter of the system. We hope this will not be confusing.

The left chirality In order to describe the integrable structures of our deformed CFT, we have to consider both chiralities at the same time. To this end, let us construct the left-chiral integrable structure of the CFT, which we will then join with the right-chiral part when moving to the massive model.

Remember that the starting point of our construction was the Feigin-Fuchs free field representation of the energy-momentum tensor (2.13), which allows us to express the right-chiral Hilbert space $\mathscr{H}_{ch}$ as a direct sum of Fock spaces $\mathscr{H}_{ch} = \bigoplus_a \mathscr{F}_{p_a}$, with the latter generated by the free action of the Heisenberg operators $\{a_{-n}\}_{n=1}^{\infty}$ on the "vacuum" $|\,p\,\rangle$, such that $P\,|\,p\,\rangle = p\,|\,p\,\rangle$ and $a_n\,|\,p\,\rangle = 0\ ,\ \forall n > 0$. We will need to repeat these steps in the left chirality, so let us introduce the following free field[24]

$$\overline{\varphi}(\overline{w}) = \mathrm{i}\overline{Q} - \mathrm{i}\frac{2\pi}{R}\overline{w}\overline{P} + \sum_{n\neq 0}\frac{\overline{a}_{-n}}{n}e^{-\mathrm{i}\frac{2\pi}{R}n\overline{w}}\ ,$$

where the operators $\left\{\overline{Q},\overline{P};\overline{a}_n\right\}_{n\neq 0}^{n\in\mathbb{Z}}$ span an Heisenberg algebra

$$[\overline{Q},\overline{P}] = \frac{\mathrm{i}}{2}\beta^2\ ,\qquad [\overline{a}_n,\overline{a}_m] = \frac{n}{2}\beta^2\delta_{n+m,0}\ .$$

The full Hilbert space can then be expressed as

$$\mathscr{H}_{\mathrm{CFT}} = \bigoplus_a \left(\mathscr{F}_{p_a}\otimes\overline{\mathscr{F}}_{-p_a}\right)\ ,$$

where $\mathscr{F}_{p_a}\otimes\overline{\mathscr{F}}_{-p_a}$ is an irreducible representation of $Vir\otimes\overline{Vir}$ with highest weights $(h(p),h(p))$ and highest-weight vector $|\,p\,\rangle\otimes\overline{|\,-p\,\rangle}$.

Now we, most simply, define the left-chiral L-operator as

$$\overline{\mathbf{L}}_j(\lambda) \doteq \pi_j\left[e^{-\pi\mathrm{i}\overline{P}H}\,\mathscr{P}\exp\left(\lambda\int_0^R d\overline{w}\left(\overline{V}_-(\overline{w})q^{\frac{H}{2}}E + \overline{V}_+(\overline{w})q^{-\frac{H}{2}}F\right)\right)\right]\ ,$$

where $\{H,E,F\}$ are $\mathcal{U}_q\left(sl(2)\right)$ generators and $\overline{V}_{\pm}(\overline{w})$ are the left-chiral vertex operators:

$$\overline{V}_{\pm}(\overline{w}) \doteq :e^{\pm 2\overline{\varphi}(\overline{w})}:\ .$$

Just as for the right-chiral L-operators they satisfy a relation with the $\mathcal{U}_q\left(sl(2)\right)$ trigonometric R-matrix which we will call "$\overline{LL}R$ relation":

$$\left(\overline{\mathbf{L}}_j(\lambda)\otimes 1\right)\left(1\otimes\overline{\mathbf{L}}_{j'}(\lambda')\right)\mathbf{R}_{j\,j'}(\lambda/\lambda') = \mathbf{R}_{j\,j'}(\lambda/\lambda')\left(1\otimes\overline{\mathbf{L}}_{j'}(\lambda')\right)\left(\overline{\mathbf{L}}_j(\lambda)\otimes 1\right)\ .$$

Notice the different ordering of this relation with respect to the corresponding one for the right chirality (2.17), which can be traced down to the minus sign in front of $\overline{P}$ in the definition of $\overline{\varphi}$. From the knowledge of the left-chiral L-operator, one can repeat exactly what has been done for the right chirality and obtain the operators $\overline{\mathbf{T}}_j$, $\overline{\mathbf{Q}}_{\pm}$, and so on. We will not go into details as these constructions are essentially identical as the ones for the right chirality.

The massive integrable structure Now that we have the L-operators of both right and left chiralities, we need to fuse them into one single object. The "vault key" holding the two pieces together will have to be the deformation parameter $\hat{\mu}^2$ which, we recall, carries a dimension $[\text{length}]^{2\beta^2-2}$. Since the spectral parameter carries a dimension of $[\lambda] = [\text{length}]^{\beta^2-1}$, the most natural way (and, as it turns out, the correct one) to couple the chiralities is

$$\mathbb{L}_j(\mu|\lambda) \doteq \mathbf{L}_j(\lambda)\overline{\mathbf{L}}_j(\mu/\lambda)\ ,$$

where $\mu\propto\hat{\mu}$. As usual, the T-operators are obtained taking the trace over the $\mathcal{U}_q\left(sl(2)\right)$ representation[25]

$$\mathbb{T}_j(\mu|\lambda) \doteq \mathrm{tr}_{\pi_j}\left(\mathbb{L}_j(\mu|\lambda)\right)\ .$$

[24] Note the appearance of the ratio $\frac{2\pi}{R}$, due to the rescaling on a cylinder with radius R.

[25] Notice the missing exponential factor in front of the L-operator. In fact $\overline{\mathbf{T}}_j(\lambda) = \mathrm{tr}_{\pi_j}\left(e^{-\pi i PH}\overline{\mathbf{L}}_j(\lambda)\right)$ and, when combining the two chiralities, the exponential factors in front of $\mathbf{L}_j$ and $\overline{\mathbf{L}}_j$ cancel each other.

The commutativity of these operators

$$[\mathbb{T}_j(\mu|\lambda), \mathbb{T}_{j'}(\mu|\lambda)] = 0 \ ,$$

is easily inferred from the RLL and $\overline{LL}R$ relations, the commutativity of the chiralities

$$[\mathbf{L}_j(\lambda), \overline{\mathbf{L}}_{j'}(\lambda')] = 0 \ ,$$

and from the unitarity of the R-matrices

$$\mathbf{R}_{j\,j'}(\lambda)\mathbf{R}_{j\,j'}(\lambda^{-1}) = 1 \ .$$

The following two properties immediately descend from the definition of $\mathbb{T}_j$ and from the properties of $\mathbf{T}_j$ and $\overline{\mathbf{T}}_j$:

- **Massless limit**: in the limit $\mu \to 0$ we recover the right and left chiral T-operators as

$$\mathbb{T}_j(\mu|\lambda) \underset{\mu\to 0}{\longrightarrow} \mathbf{T}_j(\lambda) \otimes \overline{1} \ ,$$
$$\mathbb{T}_j(\mu|\mu/\lambda) \underset{\mu\to 0}{\longrightarrow} 1 \otimes \overline{\mathbf{T}}_j(\lambda) \ ,$$

  where the tensor notation is employed here to specify that $\mathbb{T}$ acts as the identity in the left (right) chirality space.

- **Analytic properties**: the operators $\mathbb{T}$ are single valued functions of λ^2, regular everywhere except $\lambda^2 = 0, \infty$. Moreover they inherit the asymptotic behaviour of the $\mathbf{T}$ and $\overline{\mathbf{T}}$ functions:

$$\log\left(\mathbb{T}_j(\mu|\lambda)\right) \underset{\lambda\to\infty}{\sim} m_j \frac{R}{2\pi}\lambda^{1+\xi} \ ,$$

$$\log\left(\mathbb{T}_j(\mu|\lambda)\right) \underset{\lambda\to 0}{\sim} m_j \frac{R}{2\pi}\left(\frac{\mu}{\lambda}\right)^{1+\xi} \ ,$$

  where

$$m_j = \frac{\sin\left(j\pi\xi\right)}{\sin\left(\pi\frac{\xi}{2}\right)} m \ ,$$

  and m is given in (2.22). Clearly the massive T-operators have two essential singularities: one at ∞ (like the chiral operators) and the other at 0. It is natural to expect the integrals of motion (3.5-3.6) to appear in the asymptotic expansion around these singular points.

As we already said above, all the purely algebraic relations that we displayed in the previous section transfer directly here, without any change in their appearance. So the T-system reads:

$$\mathbb{T}_j(\mu|q^{\frac{1}{2}}\lambda)\mathbb{T}_j(\mu|q^{-\frac{1}{2}}\lambda) = 1 + \mathbb{T}_{j+\frac{1}{2}}(\mu|\lambda)\mathbb{T}_{j-\frac{1}{2}}(\mu|\lambda) \ ,$$

and, when q is a root of unity, it truncates to a finite system, exactly as in the massive case, which can then be recast in a Y-system, from which a TBA equation can be extracted (under some suitable analyticity assumptions).

The difference between the massive and the massless case emerges in the analytic properties of the T-operators. As we have seen just above, the $\mathbb{T}_j$ possess two singularities instead of just one, which corresponds to the two different chiralities of the massive theory. Here below we list the conjectures about the massive T-operators

- **Asymptotic expansion**: the integrals of motion (3.5-3.6) govern the asymptotic expansions of $\mathbb{T} = \mathbb{T}_{\frac{1}{2}}$ around its singular points $\lambda^2 = 0, \infty$ as follows

$$\log\left(\mathbb{T}(\mu|\lambda)\right) = m\frac{R}{2\pi}\lambda^{1+\xi} - \sum_{n=1}^{\infty} C_n \lambda^{(1-2n)(1+\xi)} \mathbb{I}_{2n-1} \ ,$$

$$\log\left(\mathbb{T}(\mu|\lambda)\right) = m\frac{R}{2\pi}\left(\frac{\mu}{\lambda}\right)^{1+\xi} - \sum_{n=1}^{\infty} C_n \left(\frac{\mu}{\lambda}\right)^{(1-2n)(1+\xi)} \overline{\mathbb{I}}_{2n-1} \ ,$$

  where the constants C_j are given in (2.23).

- **Commutation with IMs**: The massive transfer matrices commute with all the massive IMs

$$[\mathbb{T}_j(\mu|\lambda), \mathbb{I}_{2k-1}] = [\mathbb{T}_j(\mu|\lambda), \overline{\mathbb{I}}_{2k-1}] = 0 \ .$$

- **Relation between μ and $\hat{\mu}$**: the parameter entering in $\mathbb{T}$ is related to the deformation parameter $\hat{\mu}$ as follows

$$\hat{\mu}^2 = \frac{\Gamma^2\left(1-\beta^2\right)}{\pi\left(1-2\beta^2\right)\left(3\beta^2-1\right)}\left[\frac{\Gamma\left(3\beta^2\right)\Gamma\left(\beta^2\right)}{\Gamma\left(1-3\beta^2\right)\Gamma\left(1-\beta^2\right)}\right]^{\frac{1}{2}}\mu^2 \ .$$

The last of these conjectures might appear a bit outlandish, however it becomes natural when one considers $\mathbb{T}$ to actually be the transfer matrix of the sine-Gordon model. The reasons leading us to conjecture that the operators $\mathbb{T}_j$ introduced above can be interpreted as T-operators of sine-Gordon model is briefly discussed in the following subsection.

3.3 Moving on to sine-Gordon model

The quantum sine-Gordon model on a cylinder of radius R is a massive integrable QFT described by the following action

$$\mathscr{A}_{sG} = \int_0^R du \int_{-\infty}^{\infty} dv \left[\frac{1}{16\pi}\left(\partial_\mu\phi(u,v)\right)^2 + 2\tilde{\mu}\cos\left(\beta\phi(u,v)\right)\right] \ ,$$

where $\phi(w,\overline{w})$ is a scalar field, β is the coupling and $\tilde{\mu}$ is a parameter with dimensions $[\text{length}]^{2\beta^2-2}$. It possesses an infinite set of integrals of motion whose form is exactly the same as (3.5-3.6), where now T_{2k}, Θ_{2k-2}, $\overline{T}_{2k}$ and $\overline{\Theta}_{2k-2}$ are local fields of sine-Gordon model and $\hat{\mu}$ has to be replaced by $\tilde{\mu}$. The spectrum of the model contains two topologically charged particles (the soliton and the anti-soliton) with mass $\mathfrak{M}$

$$\mathfrak{M} = \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\xi}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\xi}{2}\right)}\left[\pi\tilde{\mu}\frac{\Gamma\left(\frac{1}{1+\xi}\right)}{\Gamma\left(\frac{\xi}{1+\xi}\right)}\right]^{\frac{1+\xi}{2}} \ , \qquad \xi = \frac{\beta^2}{1-\beta^2} \ , \tag{3.7}$$

and a set of neutral particles (bound-states), whose number depends on the coupling, with masses

$$\mathfrak{m}_j = 2\mathfrak{M}\sin\left(j\pi\xi\right) \ , \qquad j = \frac{1}{2}, 1, \ldots, n \text{ s.t. } n < \frac{1}{2\xi} \ .$$

The connection between sine-Gordon model and the perturbed CFTs we considered above is not evident at first. However, as it was unveiled in [43, 44] these latter can be obtained as "quantum group reductions" of the former, as we will recall very briefly. When considered in infinite volume, sine-Gordon model exhibits a symmetry with respect to the quantum group $\mathcal{U}_{\tilde{q}}\left(SL(2)\right)$, where

$$\tilde{q} = e^{\mathrm{i}\frac{\pi}{\beta^2}} \ .$$

This means that the soliton and anti-soliton transform in the two-dimensional representation of this quantum group and that the local IMs and S-matrix commute with the generators $\left\{\tilde{H}, \tilde{E}, \tilde{F}\right\}$ of the associated quantum algebra $\mathcal{U}_{\tilde{q}}\left(sl(2)\right)$. Now, the Hilbert space $\mathscr{H}_{sG}^{\infty}$ of sine-Gordon model in infinite volume, contains a subspace $\mathscr{H}_{sG}^{\infty,\,\text{singlet}}$, consisting of those states annihilated by the $\mathcal{U}_{\tilde{q}}\left(sl(2)\right)$ generators. What is remarkable is that this last Hilbert space can be interpreted as the space of states of a certain local QFT, which was called *restricted sine-Gordon model*. As it turns out, this model coincides exactly with the perturbed CFT (3.4) (considered in infinite volume!) where the parameters $\hat{\mu}$ and $\tilde{\mu}$ are related as

$$\tilde{\mu}^2 = \frac{\left(1-2\beta^2\right)\left(3\beta^2-1\right)}{\pi}\left[\frac{\Gamma^3\left(\beta^2\right)\Gamma\left(1-3\beta^2\right)}{\Gamma^3(1-\beta^2)\Gamma\left(3\beta^2\right)}\right]^{\frac{1}{2}}\hat{\mu}^2 \ .$$

Although in finite volume the quantum group symmetry breaks down, it is still possible to define singlet states and their Hilbert sub-space $\mathscr{H}_{sG}^{R,\,\text{singlet}}$ and these still allow an interpretation in terms of deformed CFTs. In particular

$$\mathscr{H}_{sG}^{R,\,\text{singlet}} \simeq \mathscr{H}_{(1,3)} \simeq \mathscr{H}_{\text{CFT}} \ .$$

It is not difficult to verify that the action of massive T-operators and their properties, defined in the previous sub-section naturally extend to the full sine-Gordon Hilbert space $\mathscr{H}_{sG}^{R}$ so that they can be interpreted as the transfer matrices of the unrestricted sine-Gordon model. In this optics, the Feigin-Fuchs fields are naturally identified with sine-Gordon one as

$$\varphi(w) = \frac{\beta}{2}\phi(w,0) \ , \qquad \overline{\varphi}(\overline{w}) = -\frac{\beta}{2}\phi(0,R-\overline{w}) \ .$$

It can moreover be shown that the truncation of the T-system happens in sine-Gordon as well, meaning that it is possible employ the methods of subsection 2.4 to obtain TBA-like equations for the ground state of the system[26]. It is as well possible to proceed as in sub-sections 2.5 and 2.6, constructing the operators $\mathbb{Q}$ and $\mathbb{A}$ and recovering the Destri-deVega equation. This path is actually more rewarding than the Y-system one, since it works for any value of the coupling β^2 (in the region $(0,1)$, as discussed for the CFTs). In order to obtain the Q-operators, one proceeds in the same exact way as for the operators $\mathbb{T}$, that is by combining the right and left-chiral L-operators in the q-oscillator representation into a single operator $\mathbb{L}_{\pm}$ and then taking its trace. We will not delve in the detail of this construction as, really, it is a simple variation on the theme of what has been done in the CFT case. We wish instead to present a different approach to the integrability structure of sine-Gordon model, which relies on a surprising and still not completely understood connection between classical and quantum worlds.

4 Sine-Gordon model and the massive ODE/IM correspondence

In this section we wish to briefly present an approach to quantum sine-Gordon[27] model in finite geometry which was proposed in [5] by S.L. Lukyanov and A.B. Zamolodchikov. This approach relies on older studies on the so-called ODE/IM correspondence [45, 46, 47] (see [48] for a review) which related the integrals of motion of certain CFTs to the spectral properties of certain ordinary differential equations (ODE). This setting was extended by Lukyanov and Zamolodchikov to the massive case [5] with the study of sine- and sinh-Gordon cases and later this method was generalised, first to the Tzitzéica-Bullough-Dodd model [49], corresponding to the affine Lie algebra $A_2^{(2)}$, then to the Toda theories associated to the affine algebras $A_n^{(1)}$ [50]. Finally, the ODE/IM was applied to the whole set Toda field theories, associated both to simply-laced [51, 52] and non-simply-laced [53, 54] algebras.

4.1 Quantum sine-Gordon T and Q operators

Before venturing in the description of the ODE/IM correspondence for the sine-Gordon model, we wish to recapitulate the properties (proved or conjectured) satisfied by the operators $\mathbb{T}$ and $\mathbb{Q}$. These properties will allow us to identify certain particular objects in the ODE side of the correspondence with the vacuum eigenvalues of T and Q operators.

Consider the quantum sine-Gordon model as defined by the Lagrangian[28]

$$\mathscr{L}_{sG} = \frac{1}{16\pi}\left[(\partial_v\phi)^2 - (\partial_u\phi)^2\right] + 2\mu\cos(\beta\phi) \ , \tag{4.1}$$

which is obviously invariant under shifts of the field $\phi \longrightarrow \phi + 2\frac{\pi}{\beta}$. As a consequence, the Hilbert space $\mathscr{H}_{sG}$ splits into orthogonal subspaces $\mathscr{H}_k$ characterised by the *quasi-momentum* k. Denote $\mathbb{U}$ the operator performing the shift of ϕ, then:

$$\mathbb{U} \ : \ \begin{array}{c} \phi \longrightarrow \phi + 2\frac{\pi}{\beta^2} \\ |\,\Phi_k\,\rangle \longrightarrow e^{2\pi i k}\,|\,\Phi_k\,\rangle \end{array} \ , \qquad \forall\,|\,\Phi_k\,\rangle \in \mathscr{H}_k \ .$$

Let us also introduce the charge and parity operators as:

$$\mathbb{C}\phi(u,v)\mathbb{C} = -\phi(u,v) \ , \qquad \mathbb{P}\phi(u,v)\mathbb{P} = \phi(-u,v) \ .$$

[26] Actually, there exists a method [25] which, in theory, allows one to recover non-linear integral equations from the Y-system for all the eigenvalues.

[27] This approach can actually be extended to sinh-Gordon model without too many difficulties.

[28] Note that here we set the parameter in the Lagrangian as μ, while before it was $\tilde{\mu}$. We hope this will not create too much confusion.

This model possesses an infinite set of T-operators $\left\{\mathbb{T}_{\frac{j}{2}}\right\}_{j=1}^{\infty}$ and two Baxter Q-operators $\mathbb{Q}_{\pm}$. Their properties, which are mostly conjectured on the basis of massless, classical and discrete limits analysis, are listed below. From this moment on we will drop the explicit dependence of $\mathbb{T}$ and $\mathbb{Q}$ on the parameter μ. We will also use the variable $\theta = \log\left(\lambda^{1+\xi}\right)$ instead of the spectral parameter.

Properties of T-operators

- **Mutual commutativity**:
$$[\mathbb{T}_j(\theta), \mathbb{T}_{j'}(\theta')] = 0 \ ,$$
- **Invariance under discrete shift**:
$$[\mathbb{T}_j(\theta), \mathbb{U}] = 0 \ ,$$
- **Invariance under charge conjugation**:
$$[\mathbb{T}_j(\theta), \mathbb{C}] = 0 \ ,$$
- **Parity conjugation**:
$$\mathbb{P}\mathbb{T}_j(\theta)\mathbb{P} = \mathbb{T}_j(-\theta) \ ,$$
- **Analytic properties**: the functions $\mathbb{T}_j(\theta)$ are entire functions of the variable θ with essential singularities at $\theta \to \pm\infty$,
- **Hermiticity**:
$$\mathbb{T}_j^{\dagger}(\theta) = \mathbb{T}_j(\theta^*) \ ,$$
- **Periodicity**:
$$\mathbb{T}_j\left(\theta + \mathrm{i}\pi(1+\xi)\right) = \mathbb{T}_j(\theta) \ ,$$
note that this property is the translation in terms of θ of property of single-valuedness of $\mathbb{T}_j$ as a function of λ^2.
- **Fusion relation (T-system)**:
$$\mathbb{T}_{\frac{1}{2}}(\theta)\mathbb{T}_j\left(\theta + \mathrm{i}\pi\xi\frac{2j+1}{2}\right) = \mathbb{T}_{j+\frac{1}{2}}\left(\theta + \mathrm{i}\pi\xi\frac{2j+2}{2}\right) + \mathbb{T}_{j-\frac{1}{2}}\left(\theta + \mathrm{i}\pi\xi\frac{2j}{2}\right) \ ,$$
or, equivalently
$$\mathbb{T}_j\left(\theta + \mathrm{i}\pi\frac{\xi}{2}\right)\mathbb{T}_j\left(\theta - \mathrm{i}\pi\frac{\xi}{2}\right) = 1 + \mathbb{T}_{j+\frac{1}{2}}(\theta)\mathbb{T}_{j-\frac{1}{2}}(\theta) \ ,$$
- **Asymptotic behaviour on the real line**:
$$\log\left(\mathbb{T}_{\frac{1}{2}}(\theta)\right) \underset{\theta\to+\infty}{\sim} \sum_{n=0}^{\infty} 2(-1)^n \sin\left(\pi\xi\frac{2n-1}{2}\right) C_n \mathbb{I}_{2n-1} e^{(1-2n)\theta} \ ,$$
$$\log\left(\mathbb{T}_{\frac{1}{2}}(\theta)\right) \underset{\theta\to-\infty}{\sim} \sum_{n=0}^{\infty} 2(-1)^n \sin\left(\pi\xi\frac{2n-1}{2}\right) C_n \overline{\mathbb{I}}_{-2n+1} e^{(2n-1)\theta} \ ,$$
where we set $\mathbb{I}_{-1} = \overline{\mathbb{I}}_{-1} \doteq \frac{R}{2\pi}$ and $C_0 = m$.

Properties of Q-operators

- **Commutativity**:
$$[\mathbb{Q}_\pm(\theta), \mathbb{T}_j(\theta')] = [\mathbb{Q}_\pm(\theta), \mathbb{Q}_\pm(\theta')] = [\mathbb{Q}_+(\theta), \mathbb{Q}_-(\theta')] = 0 \,,$$
- **Invariance under discrete shift**:
$$[\mathbb{Q}_\pm(\theta), \mathbb{U}] = 0 \,,$$
- **Charge conjugation**:
$$\mathbb{C}\mathbb{Q}_\pm(\theta)\mathbb{C} = \mathbb{Q}_\mp(\theta) \,,$$
- **Parity conjugation**:
$$\mathbb{P}\mathbb{Q}_\pm(\theta)\mathbb{P} = \mathbb{Q}_\mp(-\theta) \,,$$
- **Analytic properties**: the functions $\mathbb{Q}_\pm(\theta)$ are entire functions of the variable θ with essential singularities at $\theta \to \pm\infty$,
- **Hermiticity**:
$$\mathbb{Q}_\pm^\dagger(\theta) = \mathbb{Q}_\pm(\theta^*) \,,$$
- **Baxter T-Q relation**:
$$\mathbb{T}_{\frac{1}{2}}(\theta)\mathbb{Q}_\pm(\theta) = \mathbb{Q}_\pm(\theta + \mathrm{i}\pi\xi) + \mathbb{Q}_\pm(\theta - \mathrm{i}\pi\xi) \,,$$
- **Shift property**:
$$\mathbb{Q}_+\left(\theta + \mathrm{i}\pi(\xi+1)\right) = \mathbb{U}\mathbb{Q}_+(\theta) \,,$$
$$\mathbb{Q}_-\left(\theta + \mathrm{i}\pi(\xi+1)\right) = \mathbb{U}^{-1}\mathbb{Q}_-(\theta) \,,$$
this property, along with the T-Q relation, can be regarded as defining the Q-operators as "Bloch-wave"[29] solutions to a second order finite difference equation,
- **Quantum Wronskian**:
$$\mathbb{Q}_+\left(\theta + \mathrm{i}\pi\frac{\xi}{2}\right)\mathbb{Q}_-\left(\theta - \mathrm{i}\pi\frac{\xi}{2}\right) - \mathbb{Q}_+\left(\theta - \mathrm{i}\pi\frac{\xi}{2}\right)\mathbb{Q}_-\left(\theta + \mathrm{i}\pi\frac{\xi}{2}\right) = \mathbb{U}^{-1} - \mathbb{U} \,,$$
- **Wronskian expression of operators $\mathbb{T}_j$**:
$$\mathbb{T}_j(\theta)\left(\mathbb{U}^{-1} - \mathbb{U}\right) = \Bigg[\mathbb{Q}_+\left(\theta + \mathrm{i}\pi\xi\frac{2j+1}{2}\right)\mathbb{Q}_-\left(\theta - \mathrm{i}\pi\xi\frac{2j+1}{2}\right) + \\ - \mathbb{Q}_+\left(\theta - \mathrm{i}\pi\xi\frac{2j+1}{2}\right)\mathbb{Q}_-\left(\theta + \mathrm{i}\pi\xi\frac{2j+1}{2}\right)\Bigg] \,,$$
- **Leading asymptotic**:
$$\mathbb{Q}_+(\theta) \underset{\Re(\theta)\to\infty}{\sim} \mathbb{U}^{\pm\frac{1}{2}}\mathbb{S}^{\frac{1}{2}} \exp\left[\mathfrak{M}R\frac{e^{\theta\mp\mathrm{i}\pi\frac{\xi+1}{2}}}{4\cos\left(\pi\frac{\xi}{2}\right)}\right] \,, \qquad \theta \in H_\pm \,,$$
$$\mathbb{Q}_+(\theta) \underset{\Re(\theta)\to-\infty}{\sim} \mathbb{U}^{\pm\frac{1}{2}}\mathbb{S}^{-\frac{1}{2}} \exp\left[\mathfrak{M}R\frac{e^{-\theta\pm\mathrm{i}\pi\frac{\xi+1}{2}}}{4\cos\left(\pi\frac{\xi}{2}\right)}\right] \,, \qquad \theta \in H_\pm \,,$$
where $H_\pm \doteq \left\{\theta \in \mathbb{C} \,\middle|\, 0 < \pm\Im(\theta) < \pi(\xi+1)\right\}$ and $\mathbb{S}$ is some operator such that
$$[\mathbb{S}, \mathbb{P}] = [\mathbb{S}, \mathbb{U}] = 0 \,, \quad \mathbb{C}\mathbb{S}\mathbb{C} = \mathbb{S}^{-1} \,, \quad \mathbb{S}^\dagger = \mathbb{S} \,.$$

[29] This is a consequence of the periodicity of $\mathbb{T}_j$.

4.2 The modified sinh-Gordon equation and its linear problem

We begin our study of quantum sine-Gordon starting from an apparently far away point. Indeed let us consider the following classical partial differential equation

$$\partial\overline{\partial}\eta(z,\overline{z}) - e^{2\eta(z,\overline{z})} + p(z)p(\overline{z})e^{-2\eta(z,\overline{z})} = 0 \ , \quad p(z) = z^{2\alpha} - s^{2\alpha} \ , \tag{4.2}$$

where z and $\overline{z}$ are formal complex variables in no way related to the space-time of the quantum model. On the other hand, the real and positive parameters α and s will be, later, related to parameters of the quantum model. This equation, whose name is *modified sinh-Gordon* (MshG) equation, arise in the context of differential geometry (see e.g. [55]) where it describes the conformal metric of certain surfaces with smooth constant mean curvature immersed in $\mathbb{R}^3$. The recent interest in this equation was sparked by its appearance in the computation of gluon scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills at strong coupling [56]; these can be analysed in terms of classical strings in AdS_5 which, in turn, lead to the study of minimal surfaces, whence the MshG equation arises. This equation is integrable, as we will see, for any choice of the function p. In our case, the MshG equation (4.2) possesses an evident discrete symmetry

$$(z,\overline{z}) \longrightarrow (e^{\mathrm{i}\frac{\pi}{\alpha}}z, e^{-\mathrm{i}\frac{\pi}{\alpha}}\overline{z}) \ , \tag{4.3}$$

and we will restrict our attention to solutions which respect this symmetry. More in detail, it is not difficult to verify that there exists a family of solutions to (4.2), parametrised by the real number $l \in [-\frac{1}{2}, \frac{1}{2}]$, satisfying the following properties (here we sit on the real slice of $\mathbb{C}^2$, by setting $z = \rho e^{\mathrm{i}\phi}$ and $\overline{z} = \rho e^{-\mathrm{i}\phi}$)[30]:

1. **Periodicity**:
$$\eta(\rho,\phi) = \eta(\rho,\phi+\frac{\pi}{\alpha}) \ ,$$
or, in other words, we consider the MshG equation restricted on the cone of apex angle $\frac{\pi}{\alpha}$
$$\mathscr{C}_{\frac{\pi}{\alpha}} \doteq \left\{(\rho,\phi) \in \mathbb{R}^+ \times [-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}) \setminus \phi + \frac{\pi}{\alpha} \sim \phi\right\} \ ,$$

2. **Analyticity**: the solution $\eta(\rho,\phi)$ is single-valued, real and finite everywhere on $\mathscr{C}_{\frac{\pi}{\alpha}}$ with the sole exception of $\rho = 0, \infty$,

3. **Large-ρ asymptotics**:
$$\eta(\rho,\phi) \underset{\rho\to\infty}{\sim} \alpha \log\rho + o(1) \ ,$$

4. **Small-ρ asymptotics**:
$$\eta(\rho,\phi) \underset{\rho\to 0}{\sim} \begin{cases} 2l\ \log\rho + O(1) & |l| < \frac{1}{2} \\ \pm\log\rho + O\left(\log(-\log\rho)\right) & l = \pm\frac{1}{2} \end{cases} \ .$$

Starting from the above small-ρ asymptotics we can iteratively construct a $(z,\overline{z}) \to (0,0)$ expansion of the following form

$$\eta(z,\overline{z}) = l\ \log(z\overline{z}) + \eta_0 + \sum_{k=1}^{\infty}\gamma_k\left(z^{2\alpha k} + \overline{z}^{2\alpha k}\right) - s^{4\alpha}\frac{e^{-2\eta_0}}{(1-2l)^2}(z\overline{z})^{1-2l} + \frac{e^{2\eta_0}}{(1+2l)^2}(z\overline{z})^{1+2l} + \cdots \ ,$$

where η_0 and $\{\gamma_k\}_{k=1}^{\infty}$ are integration constants that are to be determined by imposing the properties listed above on this expansion. The utility of this expression is that it remains valid on the whole $\mathbb{C}^2$, which means that we can safely fix z to some finite value and send only $\overline{z} \to 0$, obtaining

$$\eta(z,\overline{z}) \underset{\overline{z}\to 0}{\sim} l\ \log(z\overline{z}) + \eta_0 + \gamma(z) \ , \qquad \gamma(z) \doteq \sum_{k=1}^{\infty}\gamma_k z^{2\alpha k} \ .$$

[30] Note that here ϕ denotes the argument of z and not the sine-Gordon field! Hopefully this will not cause confusion.

As hinted at above, the MshG equation is integrable and, as such, possesses a Lax pair $\{\mathscr{D}, \overline{\mathscr{D}}\}$ and an associated linear problem (from here on, we will omit the explicit dependence on the complex variables unless necessary and denote $\overline{p} \equiv p(\overline{z})$)

$$\begin{cases} \mathscr{D}\Psi = 0 & , \qquad \mathscr{D} \doteq \partial + \frac{1}{2}\partial\eta\,\sigma^3 - e^{\tilde{\theta}}\left(\sigma^+ e^{\eta} + \sigma^- p e^{-\eta}\right) \\ \overline{\mathscr{D}}\Psi = 0 & , \qquad \overline{\mathscr{D}} \doteq \overline{\partial} - \frac{1}{2}\overline{\partial}\eta\,\sigma^3 - e^{-\tilde{\theta}}\left(\sigma^- e^{\eta} + \sigma^+ \overline{p}\, e^{-\eta}\right) \end{cases} , \tag{4.4}$$

where Ψ is a $2D$ vector function and $\{\sigma^3, \sigma^\pm\}$ are the usual Pauli matrices

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

The parameter $\tilde{\theta} = \log\tilde{\lambda}$ is called spectral parameter, just as $\theta = \log\lambda$; even though the two will turn out to be one and the same thing, we prefer to denote them differently to stress the difference in their origin.

Analysis of the linear problem around $(z, \bar{z}) = 0$ Clearly the general solution to linear problem are not known, however the knowledge of the asymptotic behaviour of the function η allow us to perform a local analysis at the singular points which, as easily verified, are the same as those of MshG equation. Let us start by analysing the behaviour of solutions around zero. Notice how the linear problem (4.4) is not invariant under the discrete symmetry (4.3); instead it is invariant under a joint rotation of ϕ and $\tilde{\theta}$[31]:

$$\hat{\Omega} \ : \qquad (\phi, \tilde{\theta}) \longrightarrow (\phi + \frac{\pi}{\alpha}, \tilde{\theta} - \mathrm{i}\frac{\pi}{\alpha}) \, .$$

This property suggests that a solution Ψ of the linear problem should be represented, in the neighborhood of zero[32], by a function of $\mathrm{i}\phi + \tilde{\theta}$. Another easily verified symmetry is the parity

$$\hat{\Pi} \ : \qquad \begin{array}{l} \tilde{\theta} \longrightarrow \tilde{\theta} - \mathrm{i}\pi \\ \mathscr{D} \longrightarrow \sigma^3 \mathscr{D} \sigma^3 \\ \overline{\mathscr{D}} \longrightarrow \sigma^3 \overline{\mathscr{D}} \sigma^3 \end{array} ,$$

under which the solution should behave as

$$\hat{\Pi} \ : \qquad \Psi \longrightarrow e^{\mathrm{i}\pi\tau}\sigma^3\Psi \ , \text{ for some } \tau \in \mathbb{R} \, .$$

With these facts in mind, we define two particular solutions $\Psi_\pm(\rho, \phi|\tilde{\theta})$ to the linear problem specifying their asymptotic behaviour (note that we assume $|l| < \frac{1}{2}$)

$$\Psi_+ \underset{\rho\to 0}{\sim} \frac{1}{\sqrt{\cos(\pi l)}} \begin{pmatrix} 0 \\ e^{(\mathrm{i}\phi + \tilde{\theta})l} \end{pmatrix} , \qquad \Psi_- \underset{\rho\to 0}{\sim} \frac{1}{\sqrt{\cos(\pi l)}} \begin{pmatrix} e^{-(\mathrm{i}\phi + \tilde{\theta})l} \\ 0 \end{pmatrix} .$$

Here follows a list of easily proven properties of these two solutions

- Analyticity: the solutions $\Psi_\pm(\rho, \phi|\tilde{\theta})$ are entire functions of $\tilde{\theta}$ for any $\phi \in [-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha})$ and any $\rho > 0$,
- $\hat{\Omega}$-invariance
$$\Psi_\pm\left(\rho, \phi + \frac{\pi}{2\alpha}|\tilde{\theta} - \mathrm{i}\frac{\pi}{2\alpha}\right) = \Psi_\pm\left(\rho, \phi - \frac{\pi}{2\alpha}|\tilde{\theta} + \mathrm{i}\frac{\pi}{2\alpha}\right) ,$$
- $\hat{\Pi}$-transformation
$$\begin{aligned} \Psi_+(\rho, \phi|\tilde{\theta} \pm \mathrm{i}\pi) &= -e^{\pm\pi\mathrm{i}l}\sigma^3\Psi_+(\rho, \phi|\tilde{\theta}) \, , \\ \Psi_-(\rho, \phi|\tilde{\theta} \pm \mathrm{i}\pi) &= e^{\mp\pi\mathrm{i}l}\sigma^3\Psi_-(\rho, \phi|\tilde{\theta}) \, , \end{aligned}$$

[31] The rotation in $\tilde{\theta}$ is actually an hyperbolic rotation

[32] This argument works around zero, since this is a Fuchsian singularity of the linear problem. The same reasoning do not hold around ∞, since the presence of the potential p causes the *Stokes phenomenon* to arise. We will return on this later, but, in short words, this means that solutions to (4.4) can be represented correctly as asymptotic expansions only inside wedges of the complex plane; as one tries to move outside these, the control on the asymptotic behavior gets lost.

- Linear independence
$$\det|(\Psi_+,\Psi_-)| = -\frac{1}{\cos(\pi l)}\ ,$$
where (Ψ_+,Ψ_-) is the matrix with Ψ_+ and Ψ_- as columns. This determinant joins the role of Wronskian of solutions,
- Complex-conjugation:
$$\begin{matrix}\Psi^*_\pm(\rho,\phi|\tilde\theta) = \sigma^1\Psi_\mp(\rho,\phi|-\tilde\theta)\\ \Psi^*_\pm(\rho,0|\tilde\theta) = \Psi_\pm(\rho,0|\tilde\theta)\end{matrix}\quad,\qquad \forall\tilde\theta\in\mathbb{R}\ .$$

Large ρ analysis of the linear problem Now we move to the study of large-ρ asymptotic of solutions to the linear problem. For this task we can employ the WKB method [15] and see that there exists a solution $\Xi_-(\rho,\phi|\tilde\theta)$ which is uniquely specified by its asymptotic behaviour
$$\Xi_- \underset{\rho\to\infty}{\sim} \begin{pmatrix} e^{-\mathrm{i}\phi\frac{\alpha}{2}}\\ -e^{\mathrm{i}\phi\frac{\alpha}{2}}\end{pmatrix}\exp\left[-2\frac{\rho^{\alpha+1}}{\alpha+1}\cosh\Big(\tilde\theta+\mathrm{i}(\alpha+1)\phi\Big)\right]\ ,\quad |\phi|<\frac{\pi}{2(\alpha+1)}\ . \tag{4.5}$$
Note that this behaviour is limited to the wedge $|\phi|\le\frac{\pi}{2(\alpha+1)}$; when crossing one of the lines $\phi=\pm\frac{\pi}{2(\alpha+1)}$, the solution moves from a decaying behaviour to a growing one. This particular fact is called *Stokes phenomenon* and the lines $\phi=\pm\frac{\pi}{2(\alpha+1)}$ are known as *Stokes lines*. The reason why we have to limit the asymptotic behaviour to a given sector of the complex plane is that only there we have full control over the decaying solution. In fact, when rotating this last across a Stokes line, exponentially decaying factors might appear in the expansion without us noticing; these will then, when crossing another Stokes line, become exponentially growing, depriving us completely of the control on the asymptotic expansion. We will not delve further in this fascinating subject and direct the interested reader to the book [57], which present this subject in a modern way, underscoring its connections with the theory of transseries, Borel summability and resurgent functions.

We have seen that a basis of solutions of our linear problem is given by $\{\Psi_\pm\}$, which are entire functions of θ. This immediately means that Ξ_- as well is entire in θ and we can thus perform analytic continuation in it. In particular we can exploit the existence of the symmetry $\hat\Omega$, which allows us to generate a new solution starting from Ξ_-:
$$\Xi_+(\rho,\phi|\tilde\theta)\doteq\hat\Omega\left[\Xi_-(\rho,\phi|\tilde\theta)\right]\equiv\Xi_-\left(\rho,\phi+\frac{\pi}{\alpha}|\tilde\theta-\mathrm{i}\frac{\pi}{\alpha}\right)\ .$$
One immediately obtain the asymptotic expansion of this function
$$\Xi_+ \underset{\rho\to\infty}{\sim} -\mathrm{i}\begin{pmatrix} e^{-\mathrm{i}\phi\frac{\alpha}{2}}\\ e^{\mathrm{i}\phi\frac{\alpha}{2}}\end{pmatrix}\exp\left[2\frac{\rho^{\alpha+1}}{\alpha+1}\cosh\Big(\tilde\theta+\mathrm{i}(\alpha+1)\phi\Big)\right]\ ,\qquad |\phi|<\frac{\pi}{2(\alpha+1)}\ ,$$
which allows us to compute the determinant
$$\det|(\Xi_-,\Xi_+)| = -2\mathrm{i}\ .$$
Thus $\{\Xi_\pm\}$ is another basis of the space of solutions to the linear problem (4.4). This, however, is not the end as we can in fact repeatedly apply the transformation $\hat\Omega$ on Ξ_- and generate an infinite set of solutions
$$\Xi_n(\rho,\phi|\tilde\theta)\doteq\hat\Omega^n\left[\Xi_-(\rho,\phi|\tilde\theta)\right]\equiv\Xi_-\left(\rho,\phi+\pi\frac{n}{\alpha}|\tilde\theta-\mathrm{i}\pi\frac{n}{\alpha}\right)\ .$$
Note that these solutions can be interpreted as decaying solutions in the wedge $-\frac{\pi}{2}\frac{1+2n}{\alpha-1}<\phi<\frac{\pi}{2}\frac{1-2n}{\alpha-1}$

Spectral determinants Since the two solutions $\Psi_\pm$ are linearly independent, they form a basis of the space of solutions of the linear problem (4.4), meaning that we can expand the solution Ξ_- as
$$\Xi_-(\rho,\phi|\tilde\theta) = Q_-(\tilde\theta)\Psi_+(\rho,\phi|\tilde\theta)+Q_+(\tilde\theta)\Psi_-(\rho,\phi|\tilde\theta)\ ,$$
where the *connection coefficients*
$$Q_\pm\equiv\pm\cos(\pi l)\det|(\Xi_-,\Psi_\pm)|\ ,$$

are functions of $\tilde{\theta}$ and l only. They are also known as *spectral determinants* for the central problem of (4.4). This last denomination simply means that the zeroes of these functions are precisely the eigenvalues of the linear problem considered on functions in $L^2(0,\infty)$: those values of the spectral parameters for which the function $\Psi_\pm$ decays at infinity. The notation chosen for these functions is not random: in the following sub-section we will see how these functions can be interpreted as the ground state eigenvalues of quantum sine-Gordon Q-functions. Similarly we can exploit the existence of the basis $\{\Xi_\pm\}$ and expand the solutions Ξ_n defined above as linear combinations:

$$\Xi_n(\rho,\phi|\tilde{\theta}) = -T_{\frac{n-2}{2}}\left(\tilde{\theta}-\mathsf{i}\pi\frac{n+1}{2\alpha}\right)\Xi_-(\rho,\phi|\tilde{\theta}) + T_{\frac{n-1}{2}}\left(\tilde{\theta}-\mathsf{i}\pi\frac{n}{2\alpha}\right)\Xi_+(\rho,\phi|\tilde{\theta})\ .$$

Here too the coefficients

$$\begin{aligned} T_n\left(\tilde{\theta}-\mathsf{i}\pi\frac{2n+1}{2\alpha}\right) &= \frac{1}{2\mathsf{i}}\det|(\Xi_{2n+1},\Xi_-)| \\ T_{n+\frac{1}{2}}\left(\tilde{\theta}-\mathsf{i}\pi\frac{2n+2}{2\alpha}\right) &= \frac{1}{2\mathsf{i}}\det|(\Xi_{2n+1},\Xi_+)|\ , \end{aligned} \tag{4.6}$$

are spectral determinants, this time for the lateral problems of of our linear system. These problems consist in determining solutions to (4.4) decaying in both wedges

$$\begin{aligned} &-\frac{\pi}{2}\frac{1}{\alpha-1} < \phi < \frac{\pi}{2}\frac{1}{\alpha-1} \text{ and } -\frac{\pi}{2}\frac{4n+3}{\alpha-1} < \phi < -\frac{\pi}{2}\frac{4n+1}{\alpha-1}\ , \quad \text{for } n\in\mathbb{Z}\ , \\ &-\frac{\pi}{2}\frac{3}{\alpha-1} < \phi < \frac{\pi}{2}\frac{-1}{\alpha-1} \text{ and } -\frac{\pi}{2}\frac{4n+1}{\alpha-1} < \phi < -\frac{\pi}{2}\frac{4n-1}{\alpha-1}\ , \quad \text{for } n\in\mathbb{Z}+\frac{1}{2}\mathbb{Z}\ , \end{aligned}$$

and its particular eigenvalues corresponds to the zeroes of the functions T_j. Again, the choice of notation for these spectral determinants hints at the fact that they can be interpreted as eigenvalues of the operators $\mathbb{T}_j$, as it will be shown later.

4.3 From spectral determinants to the quantum Q-operators

Before beginning to unveil the correspondence between the linear problem (4.4) and the quantum world, let us list the properties of the spectral determinants $Q_\pm$ introduced just above

- Analyticity: $Q_\pm(\tilde{\theta})$ are entire functions of $\tilde{\theta}$, note also that the functions are defined for $l=\pm\frac{1}{2}$ by continuity
$$\lim_{l\to\pm\frac{1}{2}}\Big(Q_+(\tilde{\theta})-Q_-(\tilde{\theta})\Big)=0$$
- Quasi-periodicity:
$$Q_\pm\left(\tilde{\theta}+\mathsf{i}\pi\frac{\alpha+1}{2\alpha}\right) = e^{\pm\mathsf{i}\pi(l+\frac{1}{2})}Q_\pm\left(\tilde{\theta}-\mathsf{i}\pi\frac{\alpha+1}{2\alpha}\right)\ ,$$
- Complex conjugation:
$$Q_\pm^*(\tilde{\theta}) = Q_\pm(\tilde{\theta}^*)\ , \qquad \forall\tilde{\theta}\in\mathbb{R}\ ,$$
- Parity symmetry:
$$Q_\pm(\tilde{\theta}) = Q_\mp(-\tilde{\theta})\ , \qquad \forall\tilde{\theta}\in\mathbb{R}\ ,$$
- Quantum Wronskian:
$$Q_+\left(\tilde{\theta}+\mathsf{i}\frac{\pi}{2\alpha}\right)Q_-\left(\tilde{\theta}-\mathsf{i}\frac{\pi}{2\alpha}\right) - Q_-\left(\tilde{\theta}+\mathsf{i}\frac{\pi}{2\alpha}\right)Q_+\left(\tilde{\theta}-\mathsf{i}\frac{\pi}{2\alpha}\right) = -2\mathsf{i}\,\cos(\pi l)\ . \tag{4.7}$$

In order to proceed, it is convenient to define the following single function of two variables

$$Q(\tilde{\theta},\tilde{k}) \doteq \begin{cases} Q_+(\tilde{\theta})\Big|_{l=2\tilde{k}-\frac{1}{2}} & 0<\tilde{k}<\frac{1}{2} \\ Q_-(\tilde{\theta})\Big|_{l=-2\tilde{k}-\frac{1}{2}} & -\frac{1}{2}<\tilde{k}<0 \end{cases}\ ,$$

where $\tilde{k} = 0$ is treated by continuity. Since, obviously, $Q(\tilde{\theta}, \tilde{k}) = Q(\tilde{\theta}, \tilde{k} + 1)$, this function admits an analytic extension to all $\tilde{k} \in \mathbb{R}$. The property above become in term of this function

$$Q\left(\tilde{\theta} + \mathrm{i}\pi\frac{\alpha+1}{\alpha}, \tilde{k}\right) = e^{2\mathrm{i}\pi k} Q(\tilde{\theta}, \tilde{k}) \ ,$$

$$Q^*(\tilde{\theta}, \tilde{k}) = Q(-\tilde{\theta}, \tilde{k}) \ , \qquad Q(-\tilde{\theta}, \tilde{k}) = Q(\tilde{\theta}, -\tilde{k}) \ ,$$

$$Q\left(\tilde{\theta} + \mathrm{i}\frac{\pi}{2\alpha}, \tilde{k}\right) Q\left(\tilde{\theta} - \mathrm{i}\frac{\pi}{2\alpha}, -\tilde{k}\right) - Q\left(\tilde{\theta} - \mathrm{i}\frac{\pi}{2\alpha}, \tilde{k}\right) Q\left(\tilde{\theta} + \mathrm{i}\frac{\pi}{2\alpha}, -\tilde{k}\right) = -2i \sin\left(2\pi\tilde{k}\right) \ .$$

The similarity of the above properties with the one listed at the beginning of this section for the operators $\mathbb{Q}_\pm$ are striking, however, in order to univocally fix the function Q we still need to determine its asymptotic behaviour and the distribution of its zeroes. Thanks to the property of periodicity we can concentrate on the strip $\tilde{H} \doteq \tilde{H}_+ \cup \tilde{H}_-$ where $\tilde{H}_\pm \doteq \left\{\tilde{\theta} \in \mathbb{C} \setminus 0 < \pm\Im(\tilde{\theta}) < \pi\frac{\alpha+1}{\alpha}\right\}$. With a careful and thorough WKB analysis of the solutions of the linear problem, it is possible to establish the following behaviours

$$Q \underset{\Re(\tilde{\theta})\to\infty}{\sim} e^{\pm\mathrm{i}\pi\tilde{k}} \mathscr{S}^{\frac{1}{2}}(\tilde{k}) \exp\left[r\frac{e^{\tilde{\theta}\mp\mathrm{i}\pi\frac{\alpha+1}{2\alpha}}}{4\cos\left(\frac{\pi}{2\alpha}\right)}\right] \ , \qquad \tilde{\theta} \in H_\pm \ ,$$

$$Q \underset{\Re(\tilde{\theta})\to-\infty}{\sim} e^{\pm\mathrm{i}\pi\tilde{k}} \mathscr{S}^{-\frac{1}{2}}(\tilde{k}) \exp\left[r\frac{e^{-\tilde{\theta}\pm\mathrm{i}\pi\frac{\alpha+1}{2\alpha}}}{4\cos\left(\frac{\pi}{2\alpha}\right)}\right] \ , \qquad \tilde{\theta} \in H_\pm \ ,$$

where we introduced $r = B\, s^{1+\alpha}$ with $B = 2\sqrt{\pi}\frac{\Gamma\left(1+\frac{1}{2\alpha}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2\alpha}\right)}$ and the function

$$\mathscr{S}(\tilde{k}) \doteq \frac{\Gamma\left(2\tilde{k}\right)}{\Gamma\left(1-2\tilde{k}\right)} 2^{4\tilde{k}-1} e^{\eta_0} \ , \qquad 0 \leq \tilde{k} \leq \frac{1}{2} \ ,$$

which enjoys the symmetries

$$\mathscr{S}(\tilde{k})\mathscr{S}(-\tilde{k}) = 1 \ , \qquad \mathscr{S}(\tilde{k}+1) = \mathscr{S}(\tilde{k}) \ .$$

Now that we verified that the asymptotic of the function Q has exactly the same form of the operator $\mathbb{Q}_+$ (for $\tilde{k} > 0$), the fact that we are actually dealing with an eigenvalue of this last in disguise is becoming more than a simple suspect. What is left to do is to identify which is this specific eigenvalue by studying the pattern of the zeroes of $Q(\tilde{\theta})$: this is done again by thorough WKB analysis (remember that they are the eigenvalues of a central problem for the system (4.4)). As it turns out, for any $k \in \mathbb{R}$, these zeroes are real, simple, symmetrically disposed with respect to the origin and accumulating at the singularities $\tilde{\theta} \to \pm\infty$, which identifies the spectral determinants $Q(\tilde{\theta}, \pm\tilde{k})$ with the eigenvalues $Q_\pm^{(\mathrm{vac})}(\theta)$ of the operator $\mathbb{Q}_\pm(\theta)$ on the vacuum state with quasi-momentum k. The parameters on the two sides of this correspondence have to be identified as

$$\alpha = \frac{1}{\xi} = \frac{1}{\beta^2} - 1 \ , \qquad \tilde{k} = k \ , \tag{4.8}$$

$$r = \mathfrak{M}R \implies s = \left(\frac{R}{\pi\beta^2}\right)^{\beta^2} \left[\mu\pi\frac{\Gamma\left(1-\beta^2\right)}{\Gamma\left(\beta^2\right)}\right]^{\frac{\beta^2}{2-2\beta^2}} \ , \tag{4.9}$$

where $\mathfrak{M}$ is the soliton mass of quantum sine-Gordon (3.7) and μ is the parameter appearing in the Lagrangian (4.1).

The NLIE equation Just as we did in the previous section, we wish to use the analytic properties of the spectral determinants/Q-functions to derive a NLIE equation, as a further verification of the identification we performed. From now on we will drop the tildas on θ and k and use equivalently α, ξ or β, depending on notational reasons. Consider the following function

$$\varepsilon(\theta) \doteq \mathsf{i} \log\left[\frac{Q(\theta+\mathsf{i}\pi\xi,k)}{Q(\theta-\mathsf{i}\pi\xi,k)}\right] ,$$

with the branch of the logarithm fixed so that

$$\varepsilon(\theta) - r\frac{e^{\theta}}{2\cos\left(\pi\frac{\xi}{2}\right)} \underset{\Re(\theta)\to\infty}{\sim} -2\pi k , \qquad |\Im(\theta)| < \frac{\pi}{2} .$$

Thanks to this function we can label univocally the zeroes of Q by integers $n \in \mathbb{Z}$ in such a way that

$$\theta_n < \theta_{n+1} , \qquad \varepsilon(\theta_n) = \pi(2n+1) ,$$

where the last relation descends from the Quantum Wronskian relation. The zeroes $\{\theta_n\}_{n\in\mathbb{Z}}$ are more conveniently represented in the following form

$$e^{2\theta_n\frac{\alpha}{\alpha+1}} = \begin{cases} s^{-2\alpha}E_n(k) & , \ n \geq 0 \\ s^{2\alpha}E_{-n-1}^{-1}(-k) & , \ n < 0 \end{cases} , \quad 0 \leq k \leq \frac{1}{2} ,$$

which makes more explicit the symmetry $n \to -n-1$. The "zeroes" $\{E_n\}_{n=0}^{\infty}$ are functions of k and satisfy the following relations

$$E_n(k+1) = E_{n+1}(k) , \qquad E_0(-k-1)E_0(k) = s^{4\alpha} ,$$

and have the following asymptotic behaviour

$$E_n(\pm k) \underset{n\to\infty}{\sim} \left[\frac{2\pi}{B}(2n \pm 2k + 1)\right]^{2\frac{\alpha}{\alpha+1}} ,$$

which, again, can be obtained by thorough WKB analysis of the linear problem. We have now all the informations needed to express the function Q as a Hadamard product

$$Q(\theta,k) = \mathscr{C}(k)e^{2k\theta\frac{\alpha}{\alpha+1}} \prod_{n=0}^{\infty}\left(1 - s^{2\alpha}\frac{e^{2\theta\frac{\alpha}{\alpha+1}}}{E_n(k)}\right)\left(1 - s^{2\alpha}\frac{e^{-2\theta\frac{\alpha}{\alpha+1}}}{E_n(-k)}\right) ,$$

which converges only for $\alpha > 1$; if one wishes to extend the above product to the region $0 < \alpha \leq 1$, then a Weierstrass prime multiplier has to be added in order to regulate the divergency of the product [58]. The normalisation in front of the product satisfies the following relations

$$\mathscr{C}(k) = \mathscr{C}(-k) , \qquad \mathscr{C}(k) = -s^{-2\alpha}E_0(k)\mathscr{C}(k+1) .$$

We can finally write down the NLIE equation for the function ε by combining the Hadamard representation of the Q, its analytic properties, its asymptotic behaviour and the equation $\varepsilon(\theta_n) = \pi(2n+1)$, obtaining

$$\varepsilon(\theta) = -2\pi k + r\sinh\theta - 2\int_{-\infty}^{\infty} d\theta' G(\theta-\theta')\Im\left[\log\left(1 + e^{-\mathsf{i}\varepsilon(\theta'-i0)}\right)\right] ,$$

where we introduced the kernel

$$G(\theta) \doteq \int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\frac{\sinh\left(\pi\nu\frac{1-\alpha}{2\alpha}\right)}{2\cosh\left(\pi\frac{\nu}{2}\right)\sinh\left(\pi\frac{\nu}{2\alpha}\right)}e^{\mathsf{i}\theta\nu} .$$

As was expected, this equation coincides exactly with the NLIE equation for the ground state of quantum sine-Gordon model [39], given the identifications (4.8-4.9) are made. Once ε is known, one can then recover the function Q from the following formula

$$\log\left(Q(\theta+\mathrm{i}\pi\frac{\alpha+1}{2\alpha},k)\right)=\frac{r}{2}\frac{\cosh\theta}{\cos\frac{\pi}{2\alpha}}+\mathrm{i}\pi k+\frac{1}{2}\log\left(\mathscr{S}(k)\right)+$$
$$+2\mathrm{i}\int_{-\infty}^{\infty}d\theta'\Im\left[F(\theta-\theta'-\mathrm{i}0)\log\left(1+e^{-\mathrm{i}\varepsilon(\theta'-i0)}\right)\right]\,, \tag{4.10}$$

where the following kernel was introduced

$$F(\theta)\doteq\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\frac{e^{\mathrm{i}\nu\theta}}{4\cosh\left(\pi\frac{\nu}{2}\right)\sinh\left(\pi\frac{\nu-\mathrm{i}0}{2\alpha}\right)}\,.$$

The formula (4.10) is actually valid for $\Im(\theta)=0$ only; however it is possible to suitably modify it so that it provides Q in the whole strip H_+. The function $\mathscr{S}(k)$ can also be recovered from the solution to the NLIE equation

$$\log\left(\mathscr{S}(k)\right)=\alpha\int_{-\infty}^{\infty}\frac{d\theta}{\pi}\Im\left[\log\left(1+e^{-\mathrm{i}\varepsilon(\theta-i0)}\right)\right]\,.$$

Before closing this in-depth box we "close the circle" by recovering the integrals of motion from the asymptotic expansion of the function Q. For this task the equation (4.10) is perfect, since it allows strightforward evaluation of the large-$|\theta|$ expansion. This most simply involves writing $F(\theta)$ as an sum over his residues in the correct half plane and plugging it in the formula for the function Q. The result is

$$\log\left(Q(\theta+\mathrm{i}\pi\frac{\alpha+1}{2\alpha},k)\right)\underset{\Re(\theta)\to\pm\infty}{\sim}\frac{re^{\pm\theta}}{4\cos\left(\frac{\pi}{2\alpha}\right)}+\mathrm{i}\pi k+\frac{1}{2}\log\left(\mathscr{S}(k)\right)+$$
$$-\sum_{n=1}^{\infty}\left[\mathscr{I}_{\pm(2n-1)}e^{\mp(2n-1)\theta}-\mathscr{G}_{\pm2n}e^{\mp2n\alpha\theta}\right]\,,$$

where we defined the following objects

$$\mathscr{I}_{\pm(2n-1)}\doteq-\frac{r}{4\cos\left(\frac{\pi}{2\alpha}\right)}\delta_{n,1}\pm\frac{(-1)^{n+1}}{\sin\left(\pi\frac{2n-1}{\alpha}\right)}\int_{-\infty}^{\infty}\frac{d\theta}{\pi}\Im\left[e^{\pm(2n-1)(\theta-\mathrm{i}0)}\log\left(1+e^{-\mathrm{i}\varepsilon(\theta-\mathrm{i}0)}\right)\right]\,,$$

$$\mathscr{G}_{\pm2n}\doteq\pm\frac{\alpha(-1)^{2n}}{\cos\left(2\pi\alpha n\right)}\int_{-\infty}^{\infty}\frac{d\theta}{\pi}\Im\left[e^{\pm4\alpha n(\theta-\mathrm{i}0)}\log\left(1+e^{-\mathrm{i}\varepsilon(\theta-\mathrm{i}0)}\right)\right]\,.$$

which are related to the local and non-local integrals of motion, e.g.

$$\begin{matrix}\mathscr{I}_{2n-1}=\mathfrak{C}_nI_{2n-1}(k)\\ \mathscr{I}_{1-2n}=\mathfrak{C}_n\overline{I}_{2n-1}(k)\end{matrix}\quad,\quad\forall n>0\,,\quad\begin{matrix}\mathbb{I}_{2n-1}\left|\Phi_k^{(\mathrm{vac})}\right\rangle=I_{2n-1}\left|\Phi_k^{(\mathrm{vac})}\right\rangle\\ \overline{\mathbb{I}}_{2n-1}\left|\Phi_k^{(\mathrm{vac})}\right\rangle=\overline{I}_{2n-1}\left|\Phi_k^{(\mathrm{vac})}\right\rangle\end{matrix}\,,$$

with

$$\mathfrak{C}_n\doteq\left(-\frac{\alpha^2}{\alpha+1}\right)^{n-1}\frac{\Gamma\left(-\frac{2n-1}{2\alpha}\right)\Gamma\left((2n-1)\frac{\alpha+1}{2\alpha}\right)}{2\sqrt{\pi}n!}\left[\frac{2\mathfrak{M}\sin\left(\frac{\pi}{2\alpha}\right)}{8\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2\alpha}\right)\Gamma\left(-\frac{1}{2\alpha}\right)\right]^{1-2n}\,.$$

Similar formulae exist for the non-linear integrals of motion.

4.4 The T-functions

We wish to conclude this review by pointing out the connection of the spectral determinants (4.6) to the T-functions of Quantum sine-Gordon. In order to bring it into light, we can expand the Ξ functions entering the definition of T_j in terms of the basis $\Psi_\pm$, remembering that $\hat{\Omega}[\Psi_\pm] = \Psi_\pm$ and then use the quantum Wronskian relation (4.7). The result is, as expected

$$\begin{aligned} T_j(\theta) = \frac{\mathsf{i}}{2\cos(\pi l)} \Big[& Q_+\left(\theta + \mathsf{i}\pi\frac{2j+1}{2\alpha}\right) Q_-\left(\theta - \mathsf{i}\pi\frac{2j+1}{2\alpha}\right) + \\ & - Q_+\left(\theta - \mathsf{i}\pi\frac{2j+1}{2\alpha}\right) Q_-\left(\theta + \mathsf{i}\pi\frac{2j+1}{2\alpha}\right)\Big] \,, \\ T_{-\frac{1}{2}}(\theta) = 0 \,, \qquad & T_0(\theta) = 1 \,, \end{aligned} \tag{4.11}$$

which is precisely the Wronskian expression for T-functions of the quantum sine-Gordon model. We easily see that this relation directly implies the validity of the T-system

$$T_{\frac{1}{2}}(\theta) T_j\left(\theta + \mathsf{i}\pi\frac{2j+1}{2\alpha}\right) = T_{j-\frac{1}{2}}\left(\theta + \mathsf{i}\pi\frac{2j+2}{2\alpha}\right) + T_{j+\frac{1}{2}}\left(\theta + \mathsf{i}\pi\frac{2j}{2\alpha}\right) \,.$$

To perform the precise identification between a spectral determinant T and the k-vacuum eigenvalue of the operator $\mathbb{T}_j$, one has to analyse the analytic properties of the first and compare them with the last's ones. At this point it is not a surprise anymore to find out that these properties match exactly, allowing us a perfect identification of the results given in this section with those presented in Sec.s 2 and 3. We will not present these calculations here, but encourage the interested reader to go through them.

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References

[1] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov,
"Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz"
Commun. Math. Phys. **177** (1996), 381-398; arXiv:hep-th/9412229.

[2] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov,
"Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation"
Commun. Math. Phys. 190 (1997), 247-278; arXiv:hep-th/9604044.

[3] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov,
"Integrable Structure of Conformal Field Theory III. The Yang-Baxter relation"
Commun. Math. Phys. 200 (1999), 297-324 ; arXiv:hep-th/9805008.

[4] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov,
"Integrable quantum field theories in finite volume: Excited state energies "
Nucl. Phys. B **489** (1997), 487-531; arXiv:hep-th/9607099.

[5] S.L. Lukyanov and A.B. Zamolodchikov,
"Quantum Sine(h)-Gordon Model and Classical Integrable Equations",
JHEP **1007** (2010) 008; arXiv:1003.5333.

[6] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov,
"Infinite conformal symmetry in two-dimensional quantum field theory",
Nucl. Phys. B **241** (1984), 333–80.

[7] Al.B. Zamolodchikov,
"From Tricritical Ising to Critical Ising by Thermodynamic Bethe Ansatz",
Nucl. Phys. B **358** (1991), 524-546.

[8] P.Ginsparg,
"Applied Conformal Field Theory",
in "Fields, Strings and Critical Phenomena",
(Les Houches, Session XLIX, 1988) ed. by E. Brézin and J. Zinn Justin (1989); arXiv: hep-th/9108028.

[9] P. Di Francesco, P. Mathieu and D. Sénéchal,
"Conformal Field Theory",
Springer Verlag, 1997.

[10] E. Witten,
"Coadjoint orbits in the Virasoro Group",
Commun. Math. Phys. **114** (1988), 1.

[11] A. Pressley and G. Segal,
"Loop groups",
Clarendon Press, Oxford (1986).

[12] R. Sasaki and I. Yamanaka,
"Virasoro algebra, vertex operators, quantum Sine- Gordon and solvable Quantum Field theories",
Adv. Stud. in Pure Math. **16** (1988), 271-296.

[13] A. Torrielli,
"Classical integrability",
in A. Cagnazzo, R. Frassek, A. Sfondrini, I.M. Szécsényi, S.J. van Tongeren, editors,
"An Integrability Primer for the Gauge-gravity Correspondence",
J. Phys. **A** (2016).

[14] O. Babelon, D. Bernard and M. Talon,
"Introduction to classical integrable systems",
Cambridge University Press (2003).

[15] C.M. Bender and S.A. Orszag,
"Advanced Mathematical Methods for Scientists and Engineers I",
Springer (1999).

[16] R. M. Miura,
"Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation",
J. Math. Phys. **9** (1968), 1202–1204.

[17] N.M. Bogoliubov, A.G. Izergin and V.E. Korepin,
"Quantum Inverse Scattering Method and Correlation Functions",
Cambridge University Press (1997).

[18] B.L. Feigin and D.B. Fuchs,
"Representations of the Virasoro algebra"
in "Topology. Proceedings, Leningrad 1982", Faddeev, L.D., Mal'cev, A.A. (eds.); Lect. Notes in Math. 1060; Berlin, Heidelberg, New York Springer 1984.

[19] P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin,
"Yang-Baxter equation and representation theory",
Lett. Math. Phys. **5** (1981), 393-403.

[20] J. Fuchs,
"Affine Lie algebras and quantum groups",
Cambridge University Press (1992).

[21] A.N. Kirillov and N.Yu. Reshetikhin,
"Exact solution of the integrable XXZ Heisenberg model with arbitrary spin",
J. Phys. A **20** (1987), 1565-1585.

[22] P. Fendley, F. Lesage and H. Saleur,
"Solving 1d plasmas and 2d boundary problems using Jack polynomial and functional relations",
J. Stat. Phys. **79** (1995), 799; arXiv:hep-th/9409176.

[23] A. Kuniba, T. Nakanishi and J. Suzuki,
"Functional relations in solvable lattice models I: functional relations and representation theory",
Int. J. Mod. Phys. **A9** (1994), 5215-5266; arXiv: hep-th/9309137.

[24] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin,
"Quantum integrable systems and elliptic solutions of classical discrete nonlinear equations",
Commun. Math. Phys. **188** (1997), 267-304; arXiv: hep-th/9604080.

[25] A. Klümper and P.A. Pearce,
"Conformal weights of RSOS lattice models and their fusion hierarchies",
J. Phys. A **183** (1992), 304-350.

[26] S.J. van Tongeren,
"Introduction to the thermodynamic Bethe Ansatz",
in A. Cagnazzo, R. Frassek, A. Sfondrini, I.M. Szécsényi, S.J. van Tongeren, editors,
"An Integrability Primer for the Gauge-gravity Correspondence",
J. Phys. **A** (2016).

[27] G. Andrews, R. Baxter and J. Forrester,
"Eight-vertex SOS model and generalized Rogers-Ramanujan identities",
J. Stat. Phys. **35** (1984), 193-266.

[28] R.J. Baxter and P.A. Pearce,
"Hard hexagons: interfacial tension and correlation length",
J. Phys. A **15** (1982), 897-910.

[29] V.V. Bazhanov and N.Yu. Reshetikhin,
"Critical RSOS models and conformal theory",
Int. J. Mod. Phys. A **4** (1989), 115-142.

[30] Al.B. Zamolodchikov,
"On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories",
Phys. Lett. B **253** (1991), 391.

[31] T.R. Klassen and E. Melzer,
"Spectral flow between conformal field theories in 1 + 1 dimensions",
Nucl. Phys. B **370** (1992), 511-570.

[32] P.G.O. Freund, T.R. Klassen and E. Melzer,
"S-matrices for perturbations of certain conformal field theories"
Phys. Lett. B **229** (1989), 243-247.

[33] D. Bombardelli,
"S-matrices and integrability",
in A. Cagnazzo, R. Frassek, A. Sfondrini, I.M. Szécsényi, S.J. van Tongeren, editors,
"An Integrability Primer for the Gauge-gravity Correspondence",
J. Phys. **A** (2016).

[34] V.G. Drinfel'd,
"Quantum Groups",
in "Proceedings of the International Congress of Mathematics, Berkeley 1986, 1",
California Acad. Press (1987), 798-820.

[35] F. Loebbert,
"Lectures on Yangian symmetry",
in A. Cagnazzo, R. Frassek, A. Sfondrini, I.M. Szécsényi, S.J. van Tongeren, editors,
"An Integrability Primer for the Gauge-gravity Correspondence",
J. Phys. **A** (2016).

[36] S.M. Khoroshkin and V.N. Tolstoy,
"The uniqueness theorem for the universal R- matrix",
Lett. Math. Phys. **24** (1992), 231-244.

[37] S.M. Khoroshkin and V.N. Tolstoy,
"Universal R-matrix for quantized nontwisted affine Lie algebras",
Func. Anal. Appl. **26** (1992), 69-71, [English translation].

[38] F. Levkovich-Maslyuk,
"Introduction to Bethe Ansatz",
in A. Cagnazzo, R. Frassek, A. Sfondrini, I.M. Szécsényi, S.J. van Tongeren, editors,
"An Integrability Primer for the Gauge-gravity Correspondence",
J. Phys. **A** (2016).

[39] C. Destri and H.J. de Vega,
"Unified approach to thermodynamic Bethe Ansatz and finite size corrections for lattice models and field theories",
Nucl. Phys. B **438** (1995), 413-454; arXiv:hep-th/9407117.

[40] A. Klümper, M.T. Batchelor and P.A. Pearce
"Central charges of the 6- and 19- vertex models with twisted boundary conditions",
J. Phys A **24** (1991), 3111-3133.

[41] A.B. Zamolodchikov and Al.B Zamolodchikov,
"Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models",
Ann. Phys. **120** (1979), 253-291.

[42] A.B. Zamolodchikov,
"Integrable field theory from conformal field theory",
Adv. Stud. in Pure Math. **19** (1989), 641-674.

[43] N.Yu. Reshetikhin and F.A. Smirnov,
"Hidden quantum group symmetry and integrable perturbations of conformal field theories",
Commun. Math. Phys. **131** (1990), 157-178.

[44] D. Bernard and A. LeClair,
"Residual quantum symmetries of the restricted Sine-Gordon theories",
Nucl. Phys. B **340** (1990), 721-751.

[45] P. Dorey and R. Tateo,
"Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations",
J.Phys. A **32** (1999), L419-L425; arXiv: hep-th/9812211.

[46] V.B. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov,
"Spectral determinants for Schrodinger equation and Q operators of conformal field theory",
J.Statist.Phys. **102** (2001), 567-576; arXiv: hep-th/9812247.

[47] P. Dorey, C. Dunning, F. Gliozzi and R. Tateo,
"On the ODE/IM correspondence for minimal models",
J. Phys. A **41** (2008), 132001; arXiv: 0712.2010.

[48] P. Dorey, C. Dunning and R.Tateo,
"The ODE/IM correspondence",
J.Phys. A **40** (2007), R205; arXiv: hep-th/0703066.

[49] P. Dorey, S. Faldella, S. Negro and R. Tateo,
"The Bethe Ansatz and the Tzitzéica-Bullough-Dodd equation",
Phil. Trans. Roy. Soc. Lond. A **371** (2013), 20120052; arXiv: 1209.5517.

[50] P. Adamopoulou and C. Dunning,
"Bethe Ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories",
J. Phys. A **47** (2014), 205205; arXiv: 1401.1187.

[51] S. Negro,
"ODE/IM correspondence in Toda field theories and fermionic basis in sin(h)-Gordon model",
PhD thesis (2014); to appear.

[52] D. Masoero, A. Raimondo and D. Valeri,
"Bethe Ansatz and the Spectral Theory of affine Lie algebra-valued connections. The simply-laced case",
arXiv: 1501.07421.

[53] D. Masoero, A. Raimondo and D. Valeri,
"Bethe Ansatz and the Spectral Theory of affine Lie algebra–valued connections. The non simply-laced case",
arXiv: 1511.00895.

[54] K. Ito and C. Locke,
"ODE/IM correspondence and Bethe ansatz for affine Toda field equations",
Nucl. Phys. B **896** (2015), 763-778; arXiv: 1502.00906.

[55] A. I. Bobenko,
"All constant mean curvature tori in R^3, S^3, H^3 in terms of theta-functions",
Math. Ann. **290** (1991), 209.

[56] L.F. Alday, D. Gaiotto and J. Maldacena,
"Thermodynamic bubble ansatz",
JHEP **1109** (2011), 032; arXiv: 0911.4708.

[57] O. Costin,
"Asymptotics and Borel summability",
Chapman and Hall/CRC (2008).

[58] J. B. Conway,
"Functions of One Complex Variable I",
Springer (1995).