Q-operators for one-loop $\mathcal{N} = 4$ super Yang-Mills theory

— *in the presence of a diagonal twist*

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Abstract

We present an approach to evaluate the full operatorial Q-system of the super spin chain underlying planar $\mathcal{N} = 4$ super Yang-Mills theory at one loop in the presence of a diagonal twist. Our method is based on the oscillator construction of Q-operators. The Q-operators are built as traces over Lax operators which are degenerate solutions of the Yang-Baxter equation. For non-compact representations these Lax operators may contain multiple infinite sums that conceal the form of the resulting functions. We determine these infinite sums and calculate the matrix elements of the lowest level Q-operators. Imposing the functional relations, we then bootstrap the other Q-operators from those of the lowest level. We exemplify this approach for non-compact spin $-s$ spin chains and apply it to $\mathcal{N} = 4$ SYM at the one-loop level using the BMN vacuum as an example.

1 Q-operator construction

The Q-operator construction is based on $\mathcal{R}$-operators with infinite-dimensional auxiliary space realised by oscillators as first introduced in [1] and studied for spin chains in [2, 3, 4]. The defining Yang-Baxter equation for the $\mathcal{R}$-operators which are the building blocks for Q-operators of the $\mathcal{N} = 4$ SYM spin chain is given by

$$\mathcal{L}^{\square,\Lambda}(x-y)L_I^{\square,osc}(x)\mathcal{R}_I^{\Lambda,osc}(y) = \mathcal{R}_I^{\Lambda,osc}(y)L_I^{\square,osc}(x)\mathcal{L}^{\square,\Lambda}(x-y)\,. \tag{1}$$

Here the Lax matrix $\mathcal{L}^{\square,\Lambda}$ intertwines the Schwinger oscillator realisation $E_{ab} = \bar{\chi}_a\chi_b$ of $\mathfrak{gl}(4|4)$ with the defining fundamental one

$$\mathcal{L}^{\square,\Lambda}(z) = z + \sum_{a,b=1}^{8}(-1)^{|b|}e_{ab}\,E_{ba}\,,. \tag{2}$$

The graded oscillators satisfy $[\chi_a, \bar{\chi}_b] = \delta_{ab}$ and the indices take the values $a, b = 1, \ldots, 8$ while $|a|$ denotes the grading $|\text{fermion}| = 1$ and $|\text{boson}| = 0$. The **256** Lax operators $L_I^{\square,osc}(z)$ labelled by the set I were derived in [3]. They read

$$L_I^{\square,osc}(z) = \begin{pmatrix} (z-s_I)\delta_{ab} - (-1)^{|b|}\bar{\xi}_{a\bar{a}}\xi_{\bar{a}b} & \bar{\xi}_{a\bar{b}} \\ -(-1)^{|b|}\xi_{\bar{a}b} & \delta_{\bar{a}\bar{b}} \end{pmatrix}, \qquad \text{for} \qquad I \subseteq \{1,\ldots,8\}\,. \tag{3}$$

The oscillators $(\xi_{\bar{a}a}, \bar{\xi}_{a\bar{a}})$ satisfy the graded Heisenberg algebra $[\xi_{\bar{a}a}, \bar{\xi}_{b\bar{b}}] = \delta_{ab}\delta_{\bar{a}\bar{b}}$. The notation here is as follows: we sum over repeated indices; unbarred indices take values $a, b \in I$ while barred ones take values in its complement, $\bar{a}, \bar{b} \in \bar{I}$. The shift s_I in the spectral parameter z is introduced for convenience and defined by $2s_I = \sum_{\bar{a}\in\bar{I}}(-1)^{|\bar{a}|}$.

We can solve the Yang-Baxter equation (1) for the $\mathcal{R}$-operators that are relevant for the Q-operators of the $\mathcal{N} = 4$ SYM spin chain. One finds

$$\mathcal{R}_I(z) = e^{(-1)^{|c|+|c||\bar{c}|}\bar{\xi}_{c\bar{c}}E_{c\bar{c}}}\,\frac{\Gamma(z+1-s_I-E_{\bar{a}\bar{a}})}{\Gamma(z+1-s_I-\mathbf{C})}\,e^{-(-1)^{|d||\bar{d}|+|d|+|\bar{d}|}\xi_{\bar{d}d}E_{\bar{d}d}}\,, \tag{4}$$

where $\mathbf{C}$ is the central charge. The Q-operators are then constructed as regularised traces over monodromies built in the oscillator space $(\xi, \bar{\xi})$ and taking the tensor product in the oscillator space $E_{ab} = \bar{\chi}_a\chi_b$ as

$$\mathbf{Q}_I(z) = e^{iz\sum_{a\in I}(-1)^{|a|}\phi_a}\,\widehat{\text{str}}\,\mathcal{R}_I^{[1]}(z)\otimes\mathcal{R}_I^{[2]}(z)\otimes\ldots\otimes\mathcal{R}_I^{[L]}(z)\,. \tag{5}$$

Here we introduced eight twist fields ϕ_a and the normalised supertrace

$$\widehat{\text{str}}\,X = \frac{\text{str}\,e^{-i\sum_{a,b}(\phi_a-\phi_b)\mathbf{N}_{ab}}X}{\text{str}\,e^{-i\sum_{a,b}(\phi_a-\phi_b)\mathbf{N}_{ab}}}\,, \qquad \mathbf{N}_{ab} = \bar{\xi}_{ab}\xi_{ba}\,. \tag{6}$$

It was argued in [3] that depending on the grading the Q-operators they satisfy either the bosonic QQ-relations

$$\Delta_{ab}\mathbf{Q}_{I\cup\{a,b\}}(z)\mathbf{Q}_I(z) = \mathbf{Q}_{I\cup\{a\}}(z+\tfrac{1}{2})\mathbf{Q}_{I\cup\{b\}}(z-\tfrac{1}{2}) - \mathbf{Q}_{I\cup\{a\}}(z-\tfrac{1}{2})\mathbf{Q}_{I\cup\{b\}}(z+\tfrac{1}{2})\,, \tag{7}$$

where $|a| = |b|$ or the fermionic QQ-relations

$$\Delta_{ab}\mathbf{Q}_{I\cup\{a\}}(z)\mathbf{Q}_{I\cup\{b\}}(z) = \mathbf{Q}_{I\cup\{a,b\}}(z+\tfrac{1}{2})\mathbf{Q}_I(z-\tfrac{1}{2}) - \mathbf{Q}_{I\cup\{a,b\}}(z-\tfrac{1}{2})\mathbf{Q}_I(z+\tfrac{1}{2})\,, \tag{8}$$

where $|a| \neq |b|$. Here we defined the trigonometric prefactor $\Delta_{ab} = (-1)^{|a|}2i\sin\left(\frac{\phi_a-\phi_b}{2}\right)$.

2 Warm up: Non-compact $\mathfrak{gl}(2)$ spin chain

To exemplify the difficulties that arise when evaluating Q-operators for non-compact representations we will first have a look at the $\mathfrak{gl}(2)$ spin chain. In total we have **two** non-trivial Q-operators $\mathbf{Q}_{1,2}$. The corresponding Lax operators can be written as

$$\mathcal{R}_{\{a\}}(z) = e^{\bar{\xi}_{a\bar{a}}E_{a\bar{a}}}\frac{\Gamma(z+\frac{1}{2}-E_{\bar{a}\bar{a}})}{\Gamma(z+\frac{1}{2}-\mathbf{C})}e^{-\xi_{\bar{a}a}E_{\bar{a}a}}\,, \tag{9}$$

where $a = 1, 2$ and $\bar{a} \neq a$. For infinite-dimensional representations the Lax operator contains two infinite sums emerging from the exponential functions.

To describe non-compact spin chains with spin $-\frac{1}{2}$ we take the Jordan-Schwinger realisation and perform a particle hole transformation on the oscillators of type 1:

$$E_{ab} = \bar{\chi}_a\chi_b \qquad \text{with} \qquad (\bar{\chi}_1, \chi_1) \to (-\mathbf{b}, \bar{\mathbf{b}})\,, \qquad (\bar{\chi}_2, \chi_2) \to (\bar{\mathbf{a}}, \mathbf{a})\,. \tag{10}$$

The highest-weight state $|0\rangle$, i.e. the Fock vacuum, then satisfies

$$E_{12}|0\rangle = 0\,, \qquad E_{11}|0\rangle = \lambda_1|0\rangle = -|0\rangle\,, \qquad E_{22}|0\rangle = \lambda_2|0\rangle = 0\,, \qquad E_{21}|0\rangle \sim |m\rangle\,, \tag{11}$$

The central charge takes the value $\mathbf{C}|m\rangle = -|m\rangle$.

For the spin $-\frac{1}{2}$ representation the infinite sums in the operator $\mathcal{R}_2$ truncate but not for $\mathcal{R}_1$!

To evaluate the infinite sums we introduce the ladder decomposition of the Lax operators

$$\mathcal{R}_{\{a\}}(z) = \sum_{n=-\infty}^{+\infty}(\bar{\xi}_{a\bar{a}}E_{a\bar{a}})^{\theta(+n)|n|}\mathbb{M}_{\{a\}}(z;|n|)(-\xi_{\bar{a}a}E_{\bar{a}a})^{\theta(-n)|n|}\,, \tag{12}$$

with $\theta(-m) = \theta(0) = 0$ and $\theta(m) = 1$ for $m \in \mathbb{N}_+$. The middle part is given by an infinite sum and only depends on Cartan elements

$$\mathbb{M}_{\{a\}}(z;|n|) = \frac{1}{|n|!}\frac{\Gamma(z+\frac{1}{2}-E_{\bar{a}\bar{a}})}{\Gamma(z+\frac{1}{2}-\mathbf{C})}\,{}_3F_2(E_{\bar{a}\bar{a}}-\lambda_1, E_{\bar{a}\bar{a}}-\lambda_2+1, -\mathbf{N}_{a\bar{a}}; 1+|n|, E_{\bar{a}\bar{a}}+\frac{1}{2}-z; 1)\,, \tag{13}$$

with the $\mathfrak{gl}(2)$ weights λ_1 and λ_2. Now the matrix elements of the $\mathcal{R}$-operators can be evaluating using standard relations for Hypergeometric functions! For length $L = 2$ and magnon number $M = 0, 1, 2$ we obtain:

M	$Q_{\{1\}}(z)$	$Q_{\{2\}}(z)$
0	$2i\phi_1\left[\psi'(-z-\frac{1}{2})\right] + \mathcal{O}(\phi_1^2)$	1
1	$2i\phi_1 \times (-4) \times \left[1 + (z+1)\psi'(-z-\frac{1}{2})\right] + \mathcal{O}(\phi_1^2)$	$(z+1) + \mathcal{O}(\phi_1)$
2	$2i\phi_1 \times 9 \times \left[(z+1) + (z^2+2z+\frac{13}{12})\psi'(-z-\frac{1}{2})\right] + \mathcal{O}(\phi_1^2)$	$(z^2+2z+\frac{13}{12}) + \mathcal{O}(\phi_1)$

3 $\mathcal{N} = 4$ SYM at one-loop

To specialise our construction to $\mathcal{N} = 4$ SYM at one-loop, we first restrict to the singleton representation of $\mathfrak{u}(2,2|4)$ by choosing a grading and applying particle-hole transformation as

$$\bar{\chi}_a = (\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4, -\mathbf{b}_1, -\mathbf{b}_2)\,, \qquad \chi_a = (\mathbf{a}_1, \mathbf{a}_2, \bar{\mathbf{d}}_1, \bar{\mathbf{d}}_2, \bar{\mathbf{d}}_3, \bar{\mathbf{d}}_4, \bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2) \tag{14}$$

and requiring that the central charge vanishes, i.e. $\mathbf{C} = 0$. This gives the representation of the fields of $\mathcal{N} = 4$ SYM in terms of the oscillators typically used in the spin chain description of $\mathcal{N} = 4$ SYM at weak coupling.

3.1 The operational Q-system

Truncation of $\mathcal{R}$-operators

$\mathcal{R}_I$ has matrix elements with truncating sums if one or both of the following conditions hold:

- I contains all indices corresponding to a-type oscillators, i.e. $\{1, 2\} \subseteq I$
- I does not contain any indices corresponding to b-type oscillators , i.e. $7, 8 \notin I$

Else the matrix elements of the $\mathcal{R}$-operator will involve infinite sums.

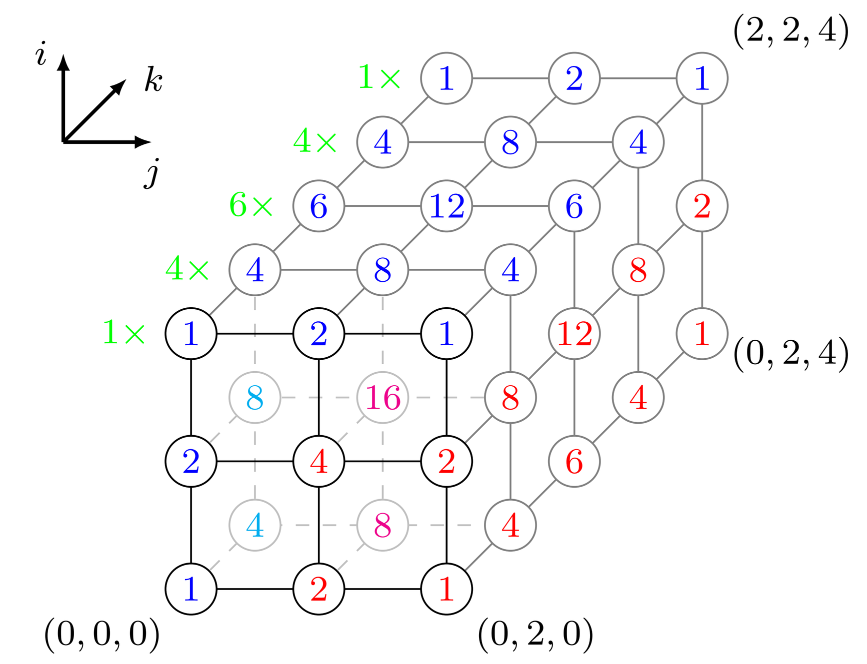

Figure 1: 3d projection of Hasse diagram

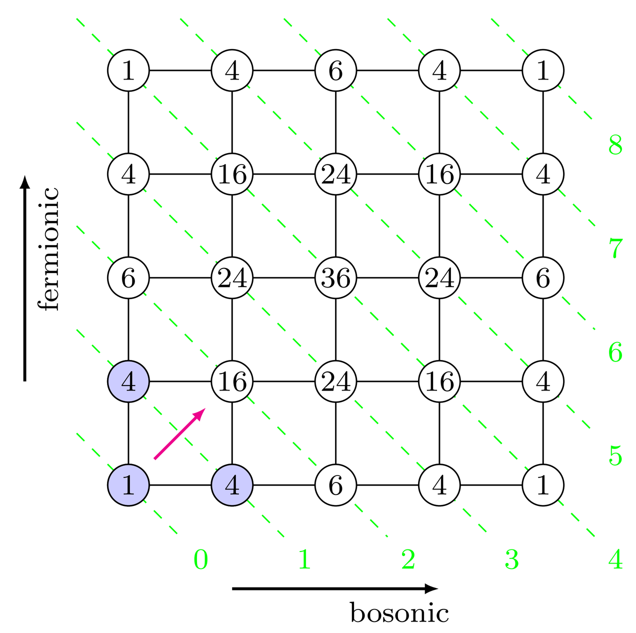

Figure 2: 2d projection of Hasse diagram

Q-operators in a 3d projection of the Hasse diagram as shown in Figure 1

- Each lattice site (i, j, k) is occupied by $\binom{2}{i} \times \binom{2}{j} \times \binom{4}{k}$ Q-operators
- In total there are $112+144= 256$ Q-operators
- The complexity of the infinite sums increases towards the lower right corner

3.2 Evaluation of Q-operators

Ladder decomposition of lowest level $\mathcal{R}$-operators

$$\mathcal{R}_{\{a\}}(z) = \sum_{\{n_{\bar{a}}\}=-\infty}^{\infty}\left[\prod_{\bar{a}}(\bar{\chi}_a\bar{\xi}_{a\bar{a}}\chi_{\bar{a}})^{\theta(+n_{\bar{a}})|n_{\bar{a}}|}\right]\mathbb{M}_{\{a\}}(z,\{\mathbf{N}\},\{n\})\left[\prod_{\bar{a}}(-(-1)^{|a|}\bar{\chi}_{\bar{a}}\xi_{\bar{a}a}\chi_a)^{\theta(-n_{\bar{a}})|n_{\bar{a}}|}\right]$$

with

$$\mathbb{M}_{\{a\}}(z,\{\mathbf{N}\},\{n\}) = \int \mathrm{d}t\; t^{-\mathbf{N}_a-1}(1-t)^{-z-1+\mathbf{C}+\frac{1}{2}\sum_{\bar{a}}(-1)^{|\bar{a}|}}\prod_{\bar{a}}\frac{1}{|n_{\bar{a}}|!}\,{}_2F_1\left({\mathbf{N}_{\bar{a}}+(-1)^{|\bar{a}|} \quad -\mathbf{N}_{a\bar{a}} \atop 1+|n_{\bar{a}}|}; (-1)^{|\bar{a}|+|a|}t\right)$$

- For truncating $\mathcal{R}$-operators, the integral above just computes a residue, while for the non-truncating ones, it is an integral over the interval $(0, 1)$
- To evaluate lowest level Q-operators the integral formula can be rewritten in terms of finite sums
- To obtain the remaining Q-operators we use QQ-relations in (7) and (8) as sketched in **Figure 2**

3.3 Lowest level Q-operators for BMN vacuum

Single-index Q-operator $\mathbf{Q}_{\{7\}}$ for the BMN vacuum $\text{tr}\,\mathcal{Z}^L$ of arbitrary length L:

$$\langle\mathcal{Z}^L|\mathbf{Q}_{\{7\}}(z)|\mathcal{Z}^L\rangle = \tau_7^{-z}\frac{(\tau_2-\tau_7)(\tau_1-\tau_7)}{(\tau_4-\tau_7)(\tau_3-\tau_7)}\left[\frac{1}{(z+\frac{1}{2})^L} + (-1)^L\frac{(\tau_2-\tau_3)(\tau_2-\tau_4)}{(\tau_1-\tau_2)\tau_2}\Phi_L^{\tau_7/\tau_2}(-z-\tfrac{1}{2}) + (-1)^L\frac{(\tau_1-\tau_3)(\tau_1-\tau_4)}{(\tau_2-\tau_1)\tau_1}\Phi_L^{\tau_7/\tau_1}(-z-\tfrac{1}{2})\right]. \tag{15}$$

Here we abbreviated the twist angles as $\tau_a = e^{-i\phi_a}$ and introduced the Lerch transcendent (Lerch zeta-function) $\Phi_\ell^\tau(z) = \sum_{k=0}^{\infty}\tau^k(k+z)^{-\ell}$.

Conclusion

- Oscillator construction applicable to non-compact spin chains
- Presented approach is valid for general $\mathfrak{u}(p,q|r)$ spin chains
- Method can efficiently be implemented on a computer

Outlook

- Q-operators for long range spin chains
- Operatorial Quantum Spectral Curve for $\mathcal{N} = 4$ SYM
- Integrable model underlying $\mathcal{N} = 4$ SYM

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